Master thesis

A study of the Bank and Weiser a posteriori estimator
for finite elements of arbitrary order

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Contents

Chapter 1

Introduction

1.1 Motivation

This work is the sequel of a precedent Master thesis on the paper [3] of R. E. Bank and A. Weiser. This paper deals with three new a posteriori error estimators for elliptic partial differential equations which are equivalent to the error (in the norm of the energy) under a binding assumption called *saturation assumption*.

The first aim of the second chapter is to describe this assumption and precise why it is a demanding assumption. The second one is to say few words about examples of problems for which this assumption is not satisfied.

In the third chapter we will work on the third a posteriori error estimator from [3], hereafter called *BW error estimator*. The aim is to prove that this estimator is equivalent to the error whithout using the saturation assumption. In order to do this, we will prove the equivalence between the residual a posteriori error estimator and BW error estimator in the norm of the energy and without the saturation assumption, firstly for piecewise linear finite elements and hopefully for finite elements of higher order. This equivalence result for piecewise linear finite elements has been already proved in a different way in [10] of R. H. Nochetto. So we hope that we could generalize it to finite elements of arbitrary order.

I want to warmly thank my directors Franz Chouly and Alexei Lozinski for their help in the redaction of this Master thesis, in the developement of the proof and more generally in my entire second year of Master.

1.2 Preliminary definitions

Our study will take place on a bounded open polygon of $\mathbb{R}^2$, denoted by Ω and of nonempty interior.

First of all, we give general definitions about Sobolev spaces. We give also the Trace theorem and the Poincaré's inequality. The definitions of Sobolev spaces can be find in [1], the Trace theorem in [9] and the Poincaré's inequality in [5].

For ω a nonempty interior open subset of $\mathbb{R}^N$ with N a positive integer and m a non negative integer, the Sobolev space $H^m(\omega)$ is defined by :

$$H^m(\omega) := \{v \in L^2(\omega) \text{ such that, } \forall \alpha \in \mathbb{N}^N,\ |\alpha| \leqslant m,\ \partial^\alpha v \in L^2(\omega)\},$$

where $\partial^\alpha v$ is the α^{th} partial derivative of v in the sense of distributions.

For each m, the space $H^m(\omega)$ is a Hilbert space for the following inner product, defined for φ and χ in $H^m(\omega)$ by :

$$(\varphi, \chi)_{m,\omega} := \sum_{\substack{\alpha \in \mathbb{N}^N \\ |\alpha| \leqslant m}} (\partial^\alpha \varphi, \partial^\alpha \chi)_\omega ,$$

of associated norm :

$$||\varphi||_{m,\omega} := \left(\sum_{\substack{\alpha \in \mathbb{N}^N \\ |\alpha| \leqslant m}} ||\partial^\alpha \varphi||^2_\omega \right)^{\frac{1}{2}} .$$

We will also need the fractional Sobolev spaces. For $0 < s < 1$, the so-called Sobolev space with fractional exponent or *fractional Sobolev space* is defined as,

$$H^s(\omega) = \left\{ v \in L^2(\omega) \text{ such that, } \frac{v(x) - v(y)}{||x-y||^{s+N/2}} \in L^2(\omega \times \omega) \right\} .$$

Furthermore, when $s > 1$ is not integer, letting $\sigma = s - [s]$ where $[s]$ is the integer part of s, $H^s(\omega)$ is defined as,

$$H^s(\omega) = \{v \in H^{[s]}(\omega) \text{ such that, } \partial^\alpha v \in H^\sigma(\omega),\ \forall \alpha,\ |\alpha| = [s]\}.$$

Let us also recall the usefull trace theorem,

Theorem 1 (Trace theorem, [9] Chap.B.3.5)**.** *Let Ω be a bounded open set and $\partial\Omega$ its polygonal boundary. Let $\gamma_0 : \mathcal{C}^0(\overline{\Omega}) \longrightarrow \mathcal{C}^0(\partial\Omega)$ map functions in $\mathcal{C}^0(\overline{\Omega})$ to their trace on $\partial\Omega$. Then, we can continuously extend to $H^1(\Omega)$ the application γ_0 in an application, again denoted γ_0, such that $\gamma_0 : H^1(\Omega) \longrightarrow H^{1/2}(\partial\Omega)$, and,*

- *γ_0 is surjective.*
- *The kernel of γ_0 is denoted $H^1_0(\Omega)$.*

The space $H_0^1(\omega)$ is a Hilbert space for the inner product of $H^1(\omega)$ but also for the inner product $(\nabla\cdot, \nabla\cdot)_\omega$ according to the following Proposition :

Proposition 1 (Poincaré's inequality, see [5] Chap. 9.4)**.** *If ω is a bounded open set of $\mathbb{R}^n$ then there exists a constant C only depending on ω such that for all v in $H_0^1(\omega)$:*

$$||v||_\omega \leqslant C||\nabla v||_\omega.$$

In particular, the application $(u, v) \longmapsto \int_\omega \nabla u \cdot \nabla v$ is a scalar product on $H_0^1(\omega)$ that induces the norm $||\nabla \cdot ||_\omega$ which is equivalent to the norm $|| \cdot ||_{H^1(\omega)}$.

A straightforward consequence of the Poincaré's inequality is that the bilinear form $(u, v) \longmapsto \int_\omega \nabla u \cdot \nabla v$ is continuous and coercive on $H_0^1(\omega)$ provided with the norm $|| \cdot ||_{H^1(\omega)}$, in other terms there exists a constant C only depending on ω such that for all v in $H_0^1(\omega)$:

$$\int_\omega (\nabla v)^2 \geqslant C||v||^2_{H^1(\omega)}. \tag{1.1}$$

1.3 Poisson problem

For the sake of simplicity, we will treat a less general problem than in [10] and only consider the Poisson problem with homogeneous Dirichlet boundary condition on a two-dimensional domain. Hopefully it could be possible to extend the following results to more general cases, like elliptic problems with Neumann boundary condition or mixed Dirichlet-Neumann boundary conditions and for instance to linear elasticity problems.

Let f be a function which belongs to $L^2(\Omega)$ and consider the following problem :

Problem 1. *Find $u : \Omega \longrightarrow \mathbb{R}$ such that :*

$$\begin{cases} -\Delta u &= f \qquad \textit{in} \quad \Omega, \\ u &= 0 \qquad \textit{on} \quad \partial\Omega, \end{cases}$$

The weak form of this problem is given by

Problem 2 (Weak form)**.** *Find a function u which belongs to $H_0^1(\Omega)$ such that for all v in $H_0^1(\Omega)$:*

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv.$$

Note : Since $(u, v) \longmapsto \int_\Omega \nabla u \cdot \nabla v$ is a inner product on $H_0^1(\Omega)$ and since $v \longmapsto \int_\Omega fv$ is a continuous linear form, then according to the Lax-Milgram theorem (see [5] Chap. 5.3) the Problem 2 has an unique solution in $H_0^1(\Omega)$ denoted by u.

1.4 Discretization

Let $\{\mathcal{T}_h\}_h$ be a family of triangulations of Ω indexed by a nonnegative real constant h called the size of $\mathcal{T}_h$ and defined as :

$$h = \max_{T \in \mathcal{T}_h} h_T,$$

where, for any T in $\mathcal{T}_h$ the real h_T is the longest edge of the triangle T.

We will suppose that the family $\{\mathcal{T}_h\}_h$ is shape regular, in other terms it exists a nonnegative real constant δ_0 such that for all $\mathcal{T}_h$ in $\{\mathcal{T}_h\}_h$ and for all T in $\mathcal{T}_h$, if we denote by ρ_T the radius of the inscribed circle of T, we have :

$$\rho_T \geqslant \delta_0 h_T.$$

Henceforth we will consider $\mathcal{T}_h$ a triangulation of $\{\mathcal{T}_h\}_h$. We will also denote by $\mathcal{E}_h$ the collection of edges of the mesh $\mathcal{T}_h$ and $\mathcal{E}_h^I$ the collection of interior edges of $\mathcal{T}_h$, in other words the edges of $\mathcal{E}_h$ which are not in $\partial\Omega$. Let us denoted by $\mathcal{N}_h$ the set of all the nodes of $\mathcal{T}_h$ and its subset $\mathcal{N}_h^I$ of interior nodes.

Let us denote by η_x the set of all the neighbouring triangles of the node x, also called *patch of* x or *star of* x. More precisely :

$$\eta_x := \bigcup_{\substack{T' \in \mathcal{T}_h \\ T' \bigcap x \neq \varnothing}} T'.$$

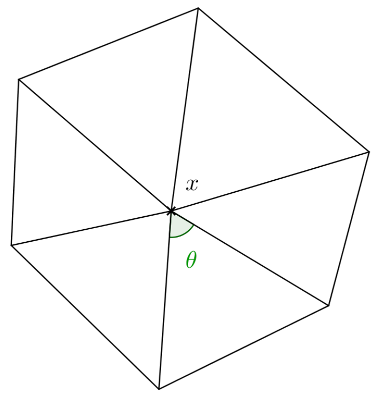
For S which can be a triangle or an edge, we denotes by $\mathrm{meas}(S)$ the 2 dimensional Lebesgue measure of S if S is a triangle and 1 dimensional Lebesgue measure if S is an edge.

Using the shape regularity of the mesh we can show that for any node x of the mesh, the number of triangles in η_x is estimated by :

$$C_0 \leqslant \mathrm{card}(\eta_x) \leqslant C_0'. \tag{1.2}$$

with $C_0 = \frac{2\pi}{\pi - \arcsin(\delta_0)}$ and $C_0' = \frac{2\pi}{\arcsin(\delta_0)}$.

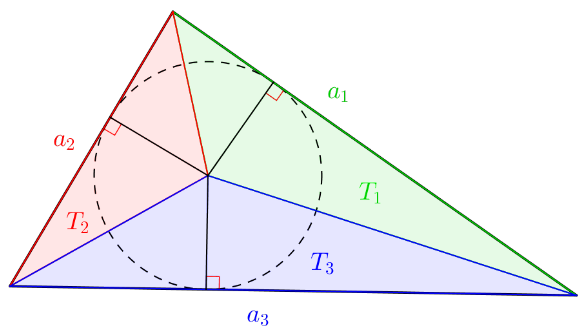
Proof. Let x be a node of the mesh. To bound the number of triangles in η_x we need to bound the angles of these triangles corresponding to the vertice x,



In order to do this we will use the following formula, for a triangle T if we denotes $\mathrm{perim}(T)$ its perimeter and since its area is given by $\mathrm{meas}(T)$ we have

$$\rho_T = \frac{2\,\mathrm{meas}(T)}{\mathrm{perim}(T)}. \tag{1.3}$$

In fact, let us split our triangle T into three triangles T_1, T_2 and T_3 and denotes the corresponding edge of T, a_1, a_2 and a_3 as follow



We have

$$\mathrm{meas}(T) = \sum_{i=1}^{3} \mathrm{meas}(T_i),$$

and for each $i \in \{1, 2, 3\}$,

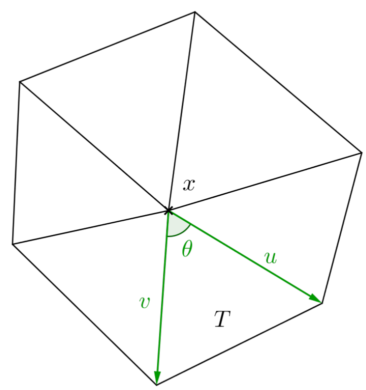
$$\mathrm{meas}(T_i) = \frac{a_i \times \rho_T}{2}.$$

So,

$$\text{meas}(T) = \sum_{i=1}^{3} \frac{a_i \times \rho_T}{2} = \frac{\rho_T}{2} \sum_{i=1}^{3} a_i = \frac{\text{perim}(T) \times \rho_T}{2}.$$

We get (1.3).

Now, for a triangle T of η_x if we denotes u and v the two vectors corresponding to the edges of T touching x, as follow



we have the following formula for the area of T,

$$\text{meas}(T) = \frac{||u|| \times ||v|| \times \sin(\theta)}{2},$$

where $||u||$ is the Euclidian norm of u. Then if we use this in (1.3) we get,

$$\begin{aligned} \rho_T &= \frac{||u|| \times ||v|| \times \sin(\theta)}{\text{perim}(T)} \\ &\leqslant \frac{h_T^2 \times \sin(\theta)}{h_T} \\ &\leqslant h_T \sin(\theta). \end{aligned}$$

Then since,

$$\sin(\theta) \geqslant \frac{\rho}{h_T} \geqslant \delta_0,$$

it implies that $\theta \in]0; \pi[$ and,

$$\arcsin(\delta_0) \leqslant \theta \leqslant \pi - \arcsin(\delta_0).$$

Now if we number θ_i for $i \in \{1, \cdots, \text{card}(\eta_x)\}$ each interior angle corresponding to a triangle of η_x we get,

$$\sum_{i=1}^{\mathrm{card}(\eta_x)} \arcsin(\delta_0) \leqslant \sum_{i=1}^{\mathrm{card}(\eta_x)} \theta_i \leqslant \sum_{i=1}^{\mathrm{card}(\eta_x)} \pi - \arcsin(\delta_0),$$

$$\mathrm{card}(\eta_x) \times \arcsin(\delta_0) \leqslant \frac{2\pi}{\mathrm{card}(\eta_x)} \leqslant \pi - \arcsin(\delta_0),$$

$$\frac{\arcsin(\delta_0)}{2\pi} \leqslant \frac{1}{\mathrm{card}(\eta_x)} \leqslant \frac{\pi - \arcsin(\delta_0)}{2\pi},$$

$$\frac{2\pi}{\pi - \arcsin(\delta_0)} \leqslant \mathrm{card}(\eta_x) \leqslant \frac{2\pi}{\arcsin(\delta_0)}.$$

□

Another consequence of the shape regularity is the local quasi-uniformity of the mesh more precisely there exists a constant $\delta_1 \geqslant 1$ only depending on δ_0 such that for all node x in $\mathcal{N}_h$ and all triangle T in η_x :

$$h_x \leqslant \delta_1 h_T, \tag{1.4}$$

where $h_x = \max_{T \in \eta_x}(h_T)$.

We also need to introduce the scaling application, denoted by $\mathcal{S}_T$ where T is a triangle of the mesh and defined in the following proposition :

Proposition 2 ([9], Chap. 1.3.2)**.** *Let $\widetilde{T}$ be the reference triangle in $\mathbb{R}^2$ of vertices $\{(0,0),(1,0),(0,1)\}$ and T a triangle of the mesh $\mathcal{T}_h$. Then there exist an affine bijection $\mathcal{S}_T$ mapping $\widetilde{T}$ on T such that :*

$$\begin{aligned} \mathcal{S}_T : \widetilde{T} &\longrightarrow T \\ \widetilde{x} &\longmapsto x = J_T\widetilde{x} + b_T, \end{aligned}$$

where J_T is a real invertible matrix 2×2 which is also the Jacobian matrix of $\mathcal{S}_T$ and b_T is a vector in $\mathbb{R}^2$.

Moreover, we can assume that for a node x of T, the application $\mathcal{S}_T$ send x on $(0,0)$.

In the same way, we define a scaling application for edges. We can set $\widetilde{E}$ the reference edge as the segment $[(0,0),(1,0)]$. Then we can define $\mathcal{S}_E$ as follow,

$$\begin{aligned} \mathcal{S}_E : \widetilde{E} &\longrightarrow E \\ \widetilde{x} &\longmapsto x = J_E\widetilde{x} + b_E, \end{aligned}$$

where J_E is a real invertible matrix 2×2 which is also the Jacobian matrix of $\mathcal{S}_E$ and b_E is a vector in $\mathbb{R}^2$.

Using these scaling applications we can set usefull results. In the sequel we will denote by n the outside normal of Ω and we will use the same notation for the outside normal of any triangle T of the mesh or for an (arbitrary) normal of an edge E. Only when it is necessary we will precise the origin of the normal, n_E for a normal from the edge E, n_T for the outside normal of a triangle T etc. On another hand, we will denote by s the 2 dimensional variable on T and $\widetilde{s}$ the 2 dimensional variable on $\widetilde{T}$. We will use the same notations for the 1 dimensional variables of E and $\widetilde{E}$. We also denote by ∇_s and $\nabla_{\widetilde{s}}$ the different gradients relative to s and $\widetilde{s}$ respectively. In the same way we will denotes by $\dfrac{\partial \cdot}{\partial n}_{|s}$ and $\dfrac{\partial \cdot}{\partial n}_{|\widetilde{s}}$ the respectives normal derivatives.

Proposition 3 ([9], Chap. 1.5.1)**.** *With notations of Proposition 2 and if $\widetilde{v}$ is a function which belongs to $H^1(\widetilde{T})$, let us define the function v on a triangle T such that :*

$$v = \widetilde{v} \circ \mathcal{S}_T^{-1}.$$

Then :

1. $Ch_T||\widetilde{v}||_{\widetilde{T}} \leqslant ||v||_T \leqslant C'h_T||\widetilde{v}||_{\widetilde{T}}$,
2. $C||\nabla_{\widetilde{s}}\widetilde{v}||_{\widetilde{T}} \leqslant ||\nabla_s v||_T \leqslant C'||\nabla_{\widetilde{s}}\widetilde{v}||_{\widetilde{T}}$,

all the constants only depends on the regularity of the mesh and are independant of h.

Proposition 4. *Let $\widetilde{v}$ be a function which belongs to $H^1(\widetilde{E})$ and v defined as follow,*

$$v = \widetilde{v} \circ \mathcal{S}_E^{-1}.$$

Then,

1. $||v||_E = h_E^{1/2}\,||\widetilde{v}||_{\widetilde{E}}$,
2. $Ch_E^{-1/2}\left|\left|\dfrac{\partial \widetilde{v}}{\partial n}_{|\widetilde{s}}\right|\right|_{\widetilde{E}} \leqslant \left|\left|\dfrac{\partial v}{\partial n}_{|s}\right|\right|_E \leqslant C'h_E^{-1/2}\left|\left|\dfrac{\partial \widetilde{v}}{\partial n}_{|\widetilde{s}}\right|\right|_{\widetilde{E}}$,

where the constants C and C' only depends on the regularity of the mesh and are independant of h.

Let us now introduce different FE spaces : for k a non negative integer, V_h (resp. V_h^f, resp. V_h^g) will be the continuous $\mathbb{P}_k$ (resp. $\mathbb{P}_{k+1}$, resp. $\mathbb{P}_{k-1}$) FE space and V_h^{disc} (resp. $V_h^{f,\text{disc}}$, resp. $V_h^{g,\text{disc}}$) their discontinuous counterparts. If $k-1=0$ we will suppose that V_h^g is the discontinuous $\mathbb{P}_0$ FE space, in other terms $V_h^g = V_h^{g,\text{disc}}$. Due to the Dirichlet boundary condition we also assume that the functions in each FE space vanishes on $\partial\Omega$.

We can now fix the discrete problem :

Problem 3 (Discrete problem)**.** *Find a function u_h which belongs to V_h such that for all v_h in V_h :*

$$\int_\Omega \nabla u_h \cdot \nabla v_h \ = \int_\Omega f v_h. \tag{1.5}$$

We will denote by e the approximation error, defined by :

$$e = u - u_h.$$

In the same way we can set the counterparts of the discrete problem in the spaces V_h^f and V_h^g,

Problem 4. *Find a function u_h^f which belongs to V_h^f such that for all v_h^f in V_h^f :*

$$\int_\Omega \nabla u_h^f \cdot \nabla v_h^f \ = \int_\Omega f v_h^f, \tag{1.6}$$

and,

Problem 5. *Find a function u_h^g which belongs to V_h^g such that for all v_h^g in V_h^g :*

$$\int_\Omega \nabla u_h^g \cdot \nabla v_h^g \ = \int_\Omega f v_h^g. \tag{1.7}$$

In the sequel we will denotes by u_h^f and u_h^g the respective solutions of these problems.

To define the estimator of Bank and Weiser we need to introduce a Lagrange-based polynomial interpolation operator $\mathcal{I}$ such that :

$$\mathcal{I} : V_h^{f,\text{disc}} \longrightarrow V_h^{g,\text{disc}},$$

and :

1. for all $v_h \in V_h^{g,\text{disc}}$, $\mathcal{I} v_h = v_h$,
2. for all $v_h \in V_h^f$, $\mathcal{I} v_h$ belongs to V_h^g ($\mathcal{I}$ preserve continuity),
3. there exists a constant C_0 depending on δ_0, k (the maximum degree of polynoms in V_h) and $\mathcal{I}$ but independant of h such that :

$$\sup_{\substack{v_h \in V_h^{f,\text{disc}} \\ v_h \neq 0}} \frac{||\nabla(\mathcal{I} v_h)||_\Omega}{||\nabla v_h||_\Omega} \leqslant C_0.$$

Let us denoted by V_h^0 the kernel of the operator $\mathcal{I}$:

$$V_h^0 = \{v_h \in V_h^{f,\mathrm{disc}},\ \mathcal{I}v_h = 0\}.$$

In order to prove the fiability and efficacity results for the residual estimator, we also need a quasi-interpolation operator for nonsmooth functions which respect the Dirichlet boundary condition. We use for this the Bernardi-Girault quasi-interpolation operator which is introduced in [4] and denoted $\mathcal{R}$. For any function v in $H_0^1(\Omega)$, its image by the Bernardi-Girault quasi-interpolant $\mathcal{R}$ belongs to V_h.

Now we can set interpolation error results corresponding at each interpolation operator :

Theorem 2 (Lagrange interpolation error, [2], Chap. 1.3.7)**.** *Let T be a triangle of $\mathcal{T}_h$ and s and t two nonnegative reals with s in $[0;1]$. Then there exists a constant $C_L > 0$ depending on the regularity of the mesh $\mathcal{T}_h$, on s and t and on the interpolant $\mathcal{I}$ but which not depend on h such that for all $v_h \in V_h^f(T)$ we have :*

$$|v_h - \mathcal{I}(v_h)|_{H^s(T)} \leqslant C_L h_T^{t-s} |v_h|_{H^t(T)}.$$

Theorem 3 (Bernardi-Girault interpolation error, [4])**.** *Let T be a triangle of $\mathcal{T}_h$ and s be a real in $[0;1]$ and t satisfy $s \leqslant t \leqslant k+1$ then there exist a constant $C_R > 0$ depending on the regularity of the mesh $\mathcal{T}_h$, on s and t and on the interpolant $\mathcal{R}$ but which is independant of h such that for all $v \in H^t(T)$ we have :*

$$|v - \mathcal{R}(v)|_{H^s(T)} \leqslant C_R h_T^{t-s} |v|_{H^t(\eta_T)}, \tag{1.8}$$

and,

$$|v - \mathcal{R}(v)|_{H^s(E)} \leqslant C_R h_E^{t-s-1/2} ||v||_{H^t(\eta_E)} \tag{1.9}$$

where E is an edge of the element T.

Chapter 2

Definitions of estimators

Let E in $\mathcal{E}_h$ be an interior edge of the mesh which is shared between two triangles T_1 and T_2, we denote by n_E an arbitrary normal of E and we name T_1 the triangle which n_E is its outward normal. For a function v piecewise continuous on the mesh $\mathcal{T}_h$ and for an edge E in $\mathcal{E}_h$, we define the jump of v on E, denoted $[\![v]\!]_E$ by :

$$[\![v]\!]_E(x) = v_{|T_2}(x) - v_{|T_1}(x).$$

If φ is a function in $L^2(\Omega)$, we define $\mathrm{osc}(\varphi)$ the oscillations of the function φ by :

$$\mathrm{osc}(\varphi) = \inf_{\varphi_h \in V_h^g} \left(\sum_{T \in \mathcal{T}_h} h_T^2 ||\varphi - \varphi_h||_T^2 \right)^{1/2}.$$

2.1 Residual a posteriori error estimator

First of all, let us give a characterization of the error e :

Proposition 5 (Error equation). *For all function v in $H_0^1(\Omega)$ we have :*

$$\int_\Omega \nabla e \cdot \nabla v = \sum_{T \in \mathcal{T}_h} F_T(v), \tag{2.1}$$

where, for all T in $\mathcal{T}_h$:

$$F_T(v) := \int_T rv \; + \frac{1}{2} \int_{\partial T} J_h v, \tag{2.2}$$

with $r := f + \Delta u_h$ and $J_h := \left[\!\!\left[\dfrac{\partial u_h}{\partial n} \right]\!\!\right]$.

Proof. Let v be a function of $H_0^1(\Omega)$. So by definition of the error e we have :

$$\begin{aligned} \int_\Omega \nabla e \cdot \nabla v &= \int_\Omega \nabla u \cdot \nabla v \; - \int_\Omega \nabla u_h \cdot \nabla v \\ &= \int_\Omega fv \; - \int_\Omega \nabla u_h \cdot \nabla v. \end{aligned} \tag{2.3}$$

Now we want to apply the Green formula to the second integral but the function u_h is not rather reguliar on Ω : we need at least $H^2(\Omega)$ and u_h is only continuous. So we have to split the integral into a sum of integrals on each triangle T of the mesh, where u_h is polynomial and fairly reguliar. Then :

$$\int_\Omega \nabla u_h \cdot \nabla v = \sum_{T\in\mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v$$

$$= \sum_{T\in\mathcal{T}_h} \left(-\int_T \Delta u_h v \ + \int_{\partial T} \frac{\partial u_h}{\partial n_T} v \right).$$

So if we replace this in (2.3):

$$\int_\Omega \nabla e \cdot \nabla v = \sum_{T\in\mathcal{T}_h} \left(\int_T (f + \Delta u_h) v \ - \int_{\partial T} \frac{\partial u_h}{\partial n_T} v \right).$$

However, since v is in $H^1_0(\Omega)$ the following sum :

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial u_h}{\partial n_T} v,$$

is only a sum over the interior edges of the mesh. Moreover, each interior edge E of the mesh is counted two times. Indeed, let us denote by T_1 and T_2 the two triangles which share E. For a function v on Ω we also denote by v_1 and v_2 the restrictions of v to T_1 and T_2. With these notations we have :

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial u_h}{\partial n_T} v = \sum_{E\in\mathcal{E}_h^I} \left(\int_E \frac{\partial u_{h,1}}{\partial n_{T_1}} v_1 \ + \int_E \frac{\partial u_{h,2}}{\partial n_{T_2}} v_2 \right).$$

Since v belongs to $H^1_0(\Omega)$, we have that $v_1 = v_2$ on E and if we use that $n_E = n_{T_1} = -n_{T_2}$ and the definition of the jump, we obtain :

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial u_h}{\partial n_T} v = \sum_{E\in\mathcal{E}_h^I} \int_E \left(\frac{\partial u_{h,1}}{\partial n_E} - \frac{\partial u_{h,2}}{\partial n_E} \right) v$$

$$= \sum_{E\in\mathcal{E}_h^I} \int_E \left(- \left[\!\!\left[\frac{\partial u_h}{\partial n} \right]\!\!\right] \right) v.$$

Finally :

$$\int_\Omega \nabla e \cdot \nabla v = \sum_{T\in\mathcal{T}_h} \left(\int_T (f + \Delta u_h) v \right) \ + \sum_{E\in\mathcal{E}_h^I} \left(\int_E \left[\!\!\left[\frac{\partial u_h}{\partial n} \right]\!\!\right] v \right),$$

and if we write the last sum as a sum over triangles T we obtain :

$$\int_{\Omega} \nabla e \cdot \nabla v = \sum_{T \in \mathcal{T}_h} \left(\int_T (f + \Delta u_h) v \; + \frac{1}{2} \int_{\partial T} \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right] v \right)$$

$$= \sum_{T \in \mathcal{T}_h} F_T(v).$$

□

Let us now define the residual a posteriori error estimator :

Definition 1 (Residual a posteriori estimator)**.** *We define the residual a posteriori error estimator, denoted by* E_{res} *as :*

$$E_{res} := \left(\sum_{T \in \mathcal{T}_h} h_T^2 ||f_h + \Delta u_h||_T^2 + \sum_{E \in \mathcal{E}_h} h_E \left|\left| \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right] \right|\right|_E^2 \right)^{1/2},$$

where $f_h \in V_h^g$ *is an approximation of the data* f *and* $\left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right]$ *is the jump of the normal derivative of* u_h *on the edges. From now on we will denotes :*

$$r_h := f_h + \Delta u_h,$$

the interior residual and called $J_h := \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right]$ *the edges residual.*

We can state the a posteriori estimation theorem :

Theorem 4. *It exists a positive constant* $C_{1,\mathrm{res}}$ *independant of* h *such that we have the following boundary :*

$$||\nabla e||_{\Omega} \leqslant C_{1,\mathrm{res}} \left(E_{res}^2 + \mathrm{osc}(f)^2 \right)^{1/2}.$$

Proof. As a first step, let us establish the Galerkin orthogonality. By the Problem 2 we have, for all v_h in $V_h \subset H_0^1(\Omega)$:

$$\int_{\Omega} \nabla u \cdot \nabla v_h \; = \int_{\Omega} f v_h,$$

and by the discrete Problem 3 :

$$\int_{\Omega} \nabla u_h \cdot \nabla v_h \; = \int_{\Omega} f v_h.$$

If we substract these two lines we obtain the Galerkin orthogonality :

$$\int_{\Omega} \nabla e \cdot \nabla v_h \; = 0. \tag{2.4}$$

Now let v be a function of $H_0^1(\Omega)$ and $\mathcal{R}v$ the interpolant of v which belongs to V_h. By (2.4) and by the error equation in Proposition 5 we have :

$$\int_\Omega \nabla e \cdot \nabla \mathcal{R}v \;=\; \sum_{T\in\mathcal{T}_h} \int_T r\mathcal{R}v \;+\; \sum_{E\in\mathcal{E}_h^I} \int_E J_h \mathcal{R}v = 0.$$

So if we substract this line to the error equation we got for all v in $H_0^1(\Omega)$:

$$\int_\Omega \nabla e \cdot \nabla v \;=\; \sum_{T\in\mathcal{T}_h} \int_T r(v-\mathcal{R}v) \;+\; \sum_{E\in\mathcal{E}_h^I} \int_E J_h (v-\mathcal{R}v),$$

and by Cauchy-Schwarz :

$$\int_\Omega \nabla e \cdot \nabla v \;\leqslant\; \sum_{T\in\mathcal{T}_h} ||r||_T\; ||v-\mathcal{R}v||_T \;+\; \sum_{E\in\mathcal{E}_h^I} ||J_h||_E\; ||v-\mathcal{R}v||_E. \tag{2.5}$$

By the Theorem 3 there exist a constant C only depending on δ_0 (the regularity of the mesh) such that for all v in $H_0^1(\Omega)$:

$$||v-\mathcal{R}v||_T \leqslant C h_T ||v||_{H^1(\eta_T)},$$

and,

$$||v-\mathcal{R}v||_E \leqslant C h_E^{1/2} ||v||_{H^1(\eta_E)}.$$

If we apply these inequalities and discrete Cauchy-Schwarz to (2.5) we obtain :

$$\begin{aligned}\int_\Omega \nabla e \cdot \nabla v &\leqslant \sum_{T\in\mathcal{T}_h} C h_T ||r||_T\; ||v||_{H^1(\eta_T)} \;+\; \sum_{E\in\mathcal{E}_h^I} C h_E^{1/2} ||J_h||_E\; ||v||_{H^1(\eta_E)} \\ &\leqslant C\left(\left(\sum_{T\in\mathcal{T}_h} ||v||^2_{H^1(\eta_T)}\right)^{1/2} \left(\sum_{T\in\mathcal{T}_h} h_T^2 ||r||_T^2\right)^{1/2}\right. \\ &\qquad \left. + \left(\sum_{E\in\mathcal{E}_h^I} ||v||^2_{H^1(\eta_E)}\right)^{1/2} \left(\sum_{E\in\mathcal{E}_h^I} h_E ||J_h||_E^2\right)^{1/2}\right).\end{aligned}$$

Let us now establish that there exists two constants C and C' only depending on δ_0, the regularity of the mesh, such that :

$$\left(\sum_{T\in\mathcal{T}_h} ||v||^2_{H^1(\eta_T)}\right)^{1/2} \leqslant C||v||_{H^1(\Omega)}, \tag{2.6}$$

and :

$$\left(\sum_{E\in\mathcal{E}_h^I} ||v||^2_{H^1(\eta_E)}\right)^{1/2} \leqslant C'||v||_{H^1(\Omega)}. \tag{2.7}$$

If we detail the left hand side of (2.6) we have :

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} ||v||^2_{H^1(\eta_T)} &= \sum_{T\in\mathcal{T}_h}\sum_{T'\in\eta_T} ||v||^2_{H^1(T')} \\
&= \sum_{T'\in\mathcal{T}_h}\sum_{\substack{T \text{ s.t.}\\ T'\in\eta_T}} ||v||^2_{H^1(T')} \\
&= \sum_{T'\in\mathcal{T}_h} \#\{T\in\mathcal{T}_h \text{ s.t. } T'\in\eta_T\}||v||^2_{H^1(T')} \\
&\leqslant C\sum_{T'\in\mathcal{T}_h} ||v||^2_{H^1(T')} \\
&\leqslant C||v||^2_{H^1(\Omega)},
\end{aligned}$$

where $C = \max_{T'\in\mathcal{T}_h}(\#\{T\in\mathcal{T}_h \text{ s.t. } T'\in\eta_T\})$. By (1.2) we can show that the constant C is bounded by another constant which only depends on δ_0.

By a similar argument we can set the inequality (2.7).

Then if we use (2.6) and (2.7), the concavity of the function square root and Proposition 1 we get :

$$\begin{aligned}
\int_\Omega \nabla e\cdot\nabla v &\leqslant C||v||_{H^1(\Omega)}\left(\left(\sum_{T\in\mathcal{T}_h} h_T^2||r||_T^2\right)^{1/2} + \left(\sum_{E\in\mathcal{E}_h^I} h_E||J_h||_E^2\right)^{1/2}\right) \\
&\leqslant C\,||\nabla v||_\Omega\left(\sum_{T\in\mathcal{T}_h} h_T^2||r||_T^2 + \sum_{E\in\mathcal{E}_h^I} h_E||J_h||_E^2\right)^{1/2}.
\end{aligned}$$

And now applying the inequality (1.1) and substituting e in place of v give :

$$||\nabla e||^2_\Omega \leqslant C||\nabla e||_\Omega\left(\sum_{T\in\mathcal{T}_h} h_T^2||r||_T^2 + \sum_{E\in\mathcal{E}_h^I} h_E||J_h||_E^2\right).$$

Finally dividing by $||\nabla e||_\Omega$, and using the triangular inequality gives :

$$||\nabla e||_\Omega \leqslant C \left(\sum_{T\in\mathcal{T}_h} h_T^2 ||r_h||_T^2 + \sum_{E\in\mathcal{E}_h^I} h_E ||J_h||_E^2 + \sum_{T\in\mathcal{T}_h} h_T^2 ||r - r_h||_T^2 \right)^{1/2}$$

$$\leqslant C \left(\sum_{T\in\mathcal{T}_h} h_T^2 ||r_h||_T^2 + \sum_{E\in\mathcal{E}_h^I} h_E ||J_h||_E^2 + \sum_{T\in\mathcal{T}_h} h_T^2 ||r - r_h||_T^2 \right)^{1/2}$$

Now notice that $r - r_h = f - f_h$ and when we take f_h such that :

$$\sum_{T\in\mathcal{T}_h} h_T^2 ||r - r_h||_T^2 = \mathrm{osc}(f)^2,$$

we finally have :

$$||\nabla e||_\Omega \leqslant C \left(E_{\mathrm{res}}^2 + \mathrm{osc}(f)^2 \right)^{1/2}.$$

□

Now let us define the bubble functions and set some usefull results in order to prove the efficiency of residual estimator.

Definition 2 (Bubble function, [2] Chap. 2.3.1)**.** *Let S be an element of the mesh $\mathcal{T}_h$ which can be a triangle, a node or an edge. We call bubble function associated to S and denoted b_S a function such that :*

- $b_S \in \mathbb{P}_l(\breve{S})$ *with* $l = 3$ *if* S *is a triangle and* $l = 2$ *if* S *is an edge,*
- $b_S = 0$ *on* $\partial\breve{S}$,
- $b_S > 0$ *in* $\overset{\circ}{\breve{S}}$,
- $b_S = O(1)$ *in* $\breve{S}$,

where $\breve{S} = S$ *if* S *is a triangle and* $\breve{S} = T_1 \bigcup T_2$ *if* S *is an edge (*T_1 *and* T_2 *are the two triangles sharing* E*). We extend* b_S *to all* Ω *by setting* $b_S \equiv 0$ *on* $\Omega\backslash\breve{S}$.

Proposition 6 ([2] Chap. 2.3.1)**.** *Let T be a triangle of $\mathcal{T}_h$ and b_T the bubble function associated. Then there exists a constant C independant of h_T such that for all v_h in $V_h(T)$:*

$$C^{-1}||v_h||_T^2 \leqslant \int_T v_h^2 b_T \ \leqslant C||v_h||_T^2, \tag{2.8}$$

and,

$$C^{-1}||v_h||_T \leqslant ||v_h b_T||_T + h_T||\nabla(v_h b_T)||_T \leqslant C||v_h||_T. \tag{2.9}$$

Proof. The proof consist in applying the norm equivalence in finite dimension on a reference triangle $\widetilde{T}$ fixed once for all and such that $h_T = 1$. Finally we prove the inequalities on each triangle T by using an application of scaling.

□

Let us now set the same theorem but for the edges of the mesh.

Proposition 7 ([2] Chap. 2.3.1)**.** *Let E be an edge of the mesh and b_E the corresponding edge bubble function. Then there exists a constant C independant of h_E such that for all function v_h of $V_h(E)$:*

$$C^{-1}||v_h||_E^2 \leqslant \int_E v_h^2 b_E \leqslant C||v_h||_E^2 \tag{2.10}$$

and,

$$h_{\breve{E}}^{-1/2}||v_h b_E||_{\breve{E}} + h_{\breve{E}}^{1/2}||\nabla(v_h b_E)||_{\breve{E}} \leqslant C||v_h||_E, \tag{2.11}$$

where $\breve{E} = T_1 \bigcup T_2$, T_1 and T_2 are the two triangles such that $T_1 \bigcap T_2 = E$.

Proof. The proof is similar to the one for the triangles.

□

Theorem 5 (Efficiency of residual estimator)**.** *There exists a constant C and C' only depending on the regularity of the mesh such that, for any triangle T of the mesh, we have :*

$$h_T||r_h||_T \leqslant C\left(||e||_{H^1(T)} + h_T\,||f - f_h||_T\right), \tag{2.12}$$

and for any interior edge E of the mesh,

$$h_{\breve{E}}^{1/2}||J_h||_E \leqslant C'\left(||e||_{H^1(\breve{E})} + ||f - f_h||_{\breve{E}}\right), \tag{2.13}$$

where $\breve{E} = T_1 \cup T_2$, T_1 and T_2 are the two triangles such that $T_1 \cap T_2 = E$ and $h_{\breve{E}} = \max_{T\in\breve{E}}(h_T)$.
These two inequalities implies the existence of $C_{2,res}$ only depending on the regularity of the mesh such that,

$$E_{res} \leqslant C_{2,res}\left(||\nabla e||_\Omega^2 + \mathrm{osc}^2(f)\right)^{1/2}. \tag{2.14}$$

Proof. We will use the Verfürth's bubble functions to prove (2.12) and (2.13).

1. **Let us start with** (2.12)**.** Let T be a triangle of $\mathcal{T}_h$ and b_T be the bubble function associated. Moreover, we consider f_h an approximation of f in the FE space V_h^g and denote $r_h := f_h + \Delta u_h$. With these definitions we have that r_h belongs to V_h, and by the Proposition 6 there exists a constant C independant of h_T and r_h such that :

$$||r_h||_T^2 \leqslant C\int_T r_h^2 b_T. \tag{2.15}$$

Let us now apply the error equation (2.1) with $v_h = r_h b_T \in V$. Since $r_h b_T$ vanishes on ∂T we have :

$$\int_T \nabla e \cdot \nabla (r_h b_T) = \int_T r r_h b_T$$

$$= \int_T r r_h b_T \; + \int_T r_h^2 b_T \; - \int_T r_h^2 b_T.$$

Then :

$$\int_T r_h^2 b_T \; = \int_T \nabla e \cdot \nabla (r_h b_T) \; + \int_T r_h b_T (r_h - r). \tag{2.16}$$

With Cauchy-Schwarz and the inequality (2.9) of Proposition 6 on the first integral in the right hand side :

$$\int_T \nabla e \cdot \nabla (r_h b_T) \leqslant ||\nabla e||_T ||\nabla (r_h b_T)||_T$$

$$\leqslant ||\nabla e||_T ||r_h b_T||_{H^1(T)}$$

$$\leqslant C h_T^{-1} ||\nabla e||_T ||r_h||_T,$$

where C is a constant independant of h_T and r_h.

Also by Cauchy-Schwarz and (2.9) of the Proposition 6 we have :

$$\int_T r_h b_T (r_h - r) \leqslant ||r_h b_T||_T ||r_h - r||_T$$

$$\leqslant C ||r_h||_T ||r_h - r||_T,$$

with C a constant independant of h_T and r_h.

We can now bound the left hand side of (2.16) :

$$\int_T r_h^2 b_T \; \leqslant C \left(||r_h||_T ||r_h - r||_T + h_T^{-1} ||\nabla e||_T ||r_h||_T \right),$$

and recalling the inequality (2.15) :

$$||r_h||_T^2 \leqslant C \left(||r_h||_T ||r_h - r||_T + h_T^{-1} ||\nabla e||_T ||r_h||_T \right),$$

then :

$$||r_h||_T \leqslant C \left(||r_h - r||_T + h_T^{-1} ||\nabla e||_T \right), \tag{2.17}$$

Finally since $r_h - r = f_h - f$,

$$||r_h||_T \leqslant C \left(h_T^{-1} ||\nabla e||_T + ||f_h - f||_T \right). \tag{2.18}$$

2. **Now let us prove** (2.13)**.** Let E be an edge in $\mathcal{E}_h$ and b_E its associated bubble function. By the error equation (5) and since b_E vanishes on $\partial\breve{E}$ we have :

$$\int_{\breve{E}} \nabla e \cdot \nabla(J_h b_E) = \int_{\breve{E}} r J_h b_E \ + \frac{1}{2}\int_E J_h^2 b_E.$$

Therefore :

$$\frac{1}{2}\int_E J_h^2 b_E = \int_{\breve{E}} \nabla e \cdot \nabla(J_h b_E) \ - \int_{\breve{E}} r J_h b_E. \tag{2.19}$$

Let us bound the first integral of the right hand side. Using Cauchy-Schwarz and the inequality (2.11) of the Proposition 7 we obtain :

$$\begin{aligned}\int_{\breve{E}} \nabla e \cdot \nabla(J_h b_E) &\leqslant ||\nabla e||_{\breve{E}} ||\nabla(J_h b_E)||_{\breve{E}} \\ &\leqslant ||\nabla e||_{\breve{E}} ||J_h b_E||_{H^1(\breve{E})} \\ &\leqslant C h_{\breve{E}}^{-1/2} ||\nabla e||_{\breve{E}} ||J_h||_E\end{aligned}$$

where C is a constant independant of $h_{\breve{E}}$ and J_h.

The second integral is also bound with Cauchy-Schwarz and (2.11) in Proposition 7 :

$$\begin{aligned}\int_{\breve{E}} r J_h b_E &\leqslant ||r||_{\breve{E}} ||J_h b_E||_{\breve{E}} \\ &\leqslant C h_{\breve{E}}^{1/2} ||r||_{\breve{E}} ||J_h||_E.\end{aligned}$$

Now if we apply these two inequalities to (2.19) :

$$\frac{1}{2}\int_E J_h^2 b_E \leqslant C\left(h_{\breve{E}}^{-1/2} ||\nabla e||_{\breve{E}} ||J_h||_E + h_{\breve{E}}^{1/2} ||r||_{\breve{E}} ||J_h||_E\right).$$

By the first inequality (2.10) in Proposition 7 we have :

$$\begin{aligned}||J_h||_E^2 &\leqslant C\int_E J_h^2 b_E \\ &\leqslant C\left(h_{\breve{E}}^{-1/2} ||\nabla e||_{\breve{E}} ||J_h||_E + h_{\breve{E}}^{1/2} ||r||_{\breve{E}} ||J_h||_E\right),\end{aligned}$$

then,

$$||J_h||_E \leqslant C\left(h_{\breve{E}}^{-1/2} ||\nabla e||_{\breve{E}} + h_{\breve{E}}^{1/2} ||r||_{\breve{E}}\right).$$

Finally if we apply the triangular inequality to (2.17) we get :

$$||r||_{\breve{E}} - ||r - r_h||_{\breve{E}} \leqslant ||r - (r - r_h)||_{\breve{E}} = ||r_h||_{\breve{E}} \leqslant C(h_{\breve{E}}^{-1} \, ||\nabla e||_{\breve{E}} + ||r - r_h||_{\breve{E}} \, ,$$

so

$$||r||_{\breve{E}} \leqslant C\left(h_{\breve{E}}^{-1}\,||\nabla e||_{\breve{E}} + ||r-r_h||_{\breve{E}}\right),$$

and

$$h_{\breve{E}}^{1/2}||r||_{\breve{E}} \leqslant C\left(h_{\breve{E}}^{-1/2}||\nabla e||_{\breve{E}} + h_{\breve{E}}^{1/2}||r-r_h||_{\breve{E}}\right).$$

Then

$$||J_h||_E \leqslant C\left(h_{\breve{E}}^{-1/2}||\nabla e||_{\breve{E}} + h_{\breve{E}}^{1/2}||r-r_h||_{\breve{E}}\right)$$

Now if we apply the local quasi-uniformity of the mesh we get the existence of a constant C only depending on δ_1 such that :

$$h_{\breve{E}} \leqslant C h_E.$$

Applying this to the precedent inequality gives :

$$h_E^{1/2}||J_h||_E \leqslant C(||\nabla e||_{\breve{E}} + h_E\,||f-f_h||_{\breve{E}}). \tag{2.20}$$

with C a constant independant of h but depending on the regularity of the mesh.

To show (2.14), on one hand we take the square of (2.18), use the convexity of square on the right hand side, sum over all triangles of the mesh and take the f_h which realize the oscillations of f to get,

$$\sum_{T\in\mathcal{T}_h} h_T^2\,||r_h||_T^2 \leqslant C(||\nabla e||_\Omega^2 + \mathrm{osc}^2(f)). \tag{2.21}$$

On the other hand we take the square of (2.20), use the convexity of square, sum over all the interior edges, bound the sums over edges of the right hand side by sums over triangles and take the f_h which realize the oscillations to obtain,

$$\begin{aligned}\sum_{E\in\mathcal{E}_h^I} h_E\,||J_h||_E^2 &\leqslant C\left(\sum_{E\in\mathcal{E}_h^I}||\nabla e||_{\breve{E}}^2 + \sum_{E\in\mathcal{E}_h^I} h_E^2\,||f-f_h||_{\breve{E}}^2\right)\\ &\leqslant C\left(||\nabla e||_\Omega^2 + \mathrm{osc}^2(f)\right)\end{aligned} \tag{2.22}$$

Adding (2.21) and (2.22) and taking the square root finally gives,

$$E_{\text{res}} \leqslant C\left(||\nabla e||_\Omega^2 + \mathrm{osc}^2(f)\right)^{1/2},$$

with a constant C which only depends on the regularity of the mesh.

$\square$

Note : It is important to notice that this previous result do not need any additionnal hypothesis for the regularity of the solution u.

2.2 Bank and Weiser a posteriori error estimator

Now we will adapt the a posteriori error estimator of Bank and Weiser (abbreviated as BW) to the Laplace equation with Dirichlet boundary condition. The remainder of this section is just a rewriting of what is done in [3], in the case of our problem.

Let us start with the equation of the error given in the Proposition 5. By Cauchy-Schwarz inequality and trace theorem we can check that for each triangle T in $\mathcal{T}_h$, the function F_T is linear and continuous. So with the Lax-Milgram theorem ([5], Chap. 5.3) we can build on the Proposition 5 to define the BW error estimator :

Definition-Proposition 1 (BW a posteriori error estimator)**.** *Let T be a triangle of $\mathcal{T}_h$. We denote by $\check{e}$ the unique solution in V_h^0 of the following problem :*

$$\int_T \nabla\check{e}\cdot\nabla v_h = F_T(v_h), \tag{2.23}$$

for all v_h in $V_h^0(T)$, and where the linear form F_T is defined in (2.2)*. We call BW a posteriori error estimator and denote E_{BW} the estimator :*

$$E_{BW} = \left(\sum_{T\in\mathcal{T}_h} ||\nabla\check{e}||_T^2\right)^{1/2}.$$

Note : The equation (2.23) will be also usefull in its global form :

$$\int_\Omega \nabla\check{e}\cdot\nabla v_h = \sum_{T\in\mathcal{T}_h}\left(\int_T r v_h\right) + \sum_{E\in\mathcal{E}_h^I}\left(\int_E J_h\{v_h\}\right), \tag{2.24}$$

for all v_h in $V_{\mathcal{T}_h}^0$.

Another equation will be usefull afterward :

Proposition 8. *For any v_h in V_h^f the following equation stand :*

$$\int_\Omega \nabla\check{e}\cdot\nabla(\mathrm{Id}-\mathcal{I})v_h = \int_\Omega r v_h + \sum_{E\in\mathcal{E}_h^I}\int_E J_h v_h. \tag{2.25}$$

Proof. If we rewrite equation (2.23) of Definition-Proposition 1 in the same way than in Theorem 4 we get

$$\int_\Omega \nabla\check{e}\cdot\nabla v_h = \int_\Omega (f+\Delta u_h)v_h + \sum_{E\in\mathcal{E}_h^I}\int_E J_h v_h, \tag{2.26}$$

for all v_h in $V_h \subset H_0^1(\Omega)$. And by Galerkin orthogonality

$$\int_\Omega \nabla\check{e}\cdot\nabla v_h = \int_\Omega (f+\Delta u_h)v_h + \sum_{E\in\mathcal{E}_h^I}\int_E J_h v_h = 0. \tag{2.27}$$

Also, if we take any v_h in V_h^f then $w_h := (\mathrm{Id} - \mathcal{I})v_h$ belongs to V_h^0 by definition of the interpolant $\mathcal{I}$. So we can use this function in the equation (2.23) and by (2.27), since $\mathcal{I}v_h$ belongs to V_h, we got :

$$\begin{aligned}\int_\Omega \nabla \check{e} \cdot \nabla w_h &= \int_\Omega (f + \Delta u_h) w_h \ + \sum_{E \in \mathcal{E}_h^I} \int_E J_h w_h \\ &= \int_\Omega (f + \Delta u_h) v_h \ + \sum_{E \in \mathcal{E}_h^I} \int_E J_h v_h,\end{aligned} \tag{2.28}$$

for all v_h in V_h^f.

□

As we did for the residual a posteriori estimator, we will give the result of efficacity and fiability of the BW estimator,

Theorem 6. *It exists a positive constant $\gamma < 1$ which depends on the regularity of the mesh, on k the degree of polynomials in V_h and on the choice of the interpolant $\mathcal{I}$ but independant of h such that,*

$$(1 - \beta^2)^{1/2} \, ||\nabla e||_\Omega \leqslant (1 - \gamma)^{-1/2} E_{BW} + C \operatorname{osc}(f), \tag{2.29}$$

and,

$$E_{BW} \leqslant (1 + C_{\tilde{e}}) \, ||\nabla e||_\Omega + C \operatorname{osc}(f), \tag{2.30}$$

where $C_{\tilde{e}} \in [0;1]$ is a constant depending on another a posteriori error estimator from [3].

Note : We need to say few things about the different constants which appear in the previous result :

- First, in the original paper [3] the autors build the fiability and efficiency of the BW estimator on the equivalence between E_{BW} and another a posteriori estimator which is equivalent to the error. This is where the constant $C_{\tilde{e}}$ come from.

- Then we can notice that we did not specify the origin of the constant β. In fact, this constant come from the crucial saturation assumption that we will describe in the next chapter.

- Finally we can ask the question of the *asymptotic exactness* of the BW estimator, in other words the fact of having $\lim_{h \to 0} E_{\mathrm{BW}} = ||\nabla e||_\Omega$. We could deduce this property from the precedent Theorem, only if the constants $(1 - \beta^2)$, $(1 - \gamma)$ and $(1 + C_{\tilde{e}})$ tend to 1 when h go to zero. But as we said in the theorem, the constant γ do not depends on h. Then we can not deduce the asymptotic exactness of the BW estimator from the previous result. However we can always hope that when h tends to zero, the BW estimator get really close to the true error.

Before talking of this assumption, we need to prove the slight changing we have made in Theorem 6 with respect to the original result.

Proof. The proof of this result is given in [3] and applying it to the Poisson problem is straight. We only exchange the second term of the right hand side in the original result with the oscillations of f and we need to prove that we can do this exchange. The originals inequations state as follow,

$$(1-\beta^2)^{1/2} \left|\left|\nabla e\right|\right|_\Omega \leqslant (1-\gamma)^{-1/2} E_{\mathrm{BW}} + C_0 \left|\left|\nabla e_{V_h}\right|\right|_\Omega ,$$

and,

$$E_{\mathrm{BW}} \leqslant (1 + C_{\tilde{e}}) \left|\left|\nabla e\right|\right|_\Omega + C_0 \left|\left|\nabla e_{V_h}\right|\right|_\Omega ,$$

where e_{V_h} is the difference between the approximation u_h and U which is the *computed approximation* of u, in other words in U we include all the approximations that the computer need to do to approach u. In our case we will assume that U is the solution in V_h of the following equation,

$$\int_\Omega \nabla U \cdot \nabla v_h = \int_\Omega f_h v_h, \quad \forall v_h \in V_h, \tag{2.31}$$

where f_h is an approximation of f which belongs to V_h^g.

So now, we need to prove the below inequality,

$$\left|\left|\nabla e_{V_h}\right|\right|_\Omega = \left|\left|\nabla(u_h - U)\right|\right| \leqslant C \operatorname{osc}(f),$$

with a constant C independ of h.

If we restrict the two equations (1.5) and (2.31) to a triangle T of the mesh and substract them we get,

$$\int_T \nabla(u_h - U) \cdot \nabla v_h = \int_T (f - f_h) v_h.$$

Then, by Cauchy-Schwarz inequality,

$$\int_T \nabla(u_h - U) \cdot \nabla v_h \leqslant \left|\left|f - f_h\right|\right|_T \left|\left|v_h\right|\right|_T . \tag{2.32}$$

By the norm equivalence in finite dimension and a scaling argument (Proposition 3), we have the existence of a constant C independant of h such that for every v_h in V_h,

$$\left|\left|v_h\right|\right|_T \leqslant C h_T \left|\left|\nabla v_h\right|\right|_T .$$

Now if we take $v_h = u_h - U$ in the precedent inequation and in (2.32) we get,

$$\left|\left|\nabla(u_h - U)\right|\right|^2 \leqslant C h_T \left|\left|f - f_h\right|\right|_T \left|\left|\nabla(u_h - U)\right|\right| .$$

Dividing by $\left|\left|\nabla(u_h - U)\right|\right|$, taking the square of the result and summing on all the triangles of the mesh gives,

$$\sum_{T\in\mathcal{T}_h} ||\nabla(u_h - U)||_T^2 \leqslant C \sum_{T\in\mathcal{T}_h} h_T^2 \, ||f - f_h||_T^2 \, .$$

then,

$$\left(\sum_{T\in\mathcal{T}_h} ||\nabla(u_h - U)||_T^2\right)^{1/2} \leqslant C \left(\sum_{T\in\mathcal{T}_h} h_T^2 \, ||f - f_h||_T^2\right)^{1/2} .$$

Finally if we take f_h which realize the oscillations of f,

$$||\nabla(u_h - U)||_\Omega = ||\nabla e_{V_h}|| \leqslant C \operatorname{osc}(f).$$

□

2.3 About the saturation assumption of Bank and Weiser

The original paper [3] is mainly based on a conjecture called *saturation assumption.* In [3] this assumption is given in the following form (with our notations),

Hypothesis 1. *It exists $\beta = \beta(h)$ a real valued function such that $\lim_{h\to 0} \beta = 0$ and such that we have the following boundary :*

$$\left\|\nabla(u - u_h^f)\right\|_\Omega^2 + \sum_{E\in\mathcal{E}_h^I} \left(\left\| h_E^{1/2} \left\{ \frac{\partial \left(u - u_h^f\right)}{\partial n} \right\} \right\|_E^2\right) \leqslant \beta^2 \, ||\nabla(u - u_h)||_\Omega^2 \, . \tag{2.33}$$

Then, the constant β of this assumption is the one used in Theorem 6. The inequality of this assumption means that the approximation of u by u_h^f is better than that by u_h. To clarify this note we give a simpler form of the saturation assumption,

Hypothesis 2 (Saturation assumption)**.** *It exists $\alpha = \alpha(h)$ a real valued function such that $\lim_{h\to 0} \alpha = 0$ and such that we have the following boundary,*

$$\left\|\nabla(u - u_h^f)\right\|_\Omega \leqslant \alpha \, ||\nabla(u - u_h)||_\Omega \, . \tag{2.34}$$

Proof. To replace the saturation assumption of Bank-Weiser by its simpler form we need to prove that it exists a constant C independant of h such that,

$$\sum_{E\in\mathcal{E}_h^I} \left(\left\| h_E^{1/2} \left\{ \frac{\partial \left(u - u_h^f\right)}{\partial n} \right\} \right\|_E^2\right) \leqslant C \left\|\nabla(u - u_h^f)\right\|_\Omega^2 . \tag{2.35}$$

We can prove this using the trace theorem, the Poincaré's inequality and the propositions of scaling. Let us start with $\widetilde{T}$ the reference triangle and $\widetilde{w}$ a function of $H_0^1(\widetilde{T})$. By Theorem 1 we have,

$$||\widetilde{w}||_{H^{1/2}(\partial\widetilde{T})} \leqslant C\,||\widetilde{w}||_{H^1(\widetilde{T})}\,. \tag{2.36}$$

Using Poincaré inequality of Proposition 1 on the right hand side gives a constant C which only depends on $\widetilde{T}$ such that,

$$||\widetilde{w}||_{H^{1/2}(\partial\widetilde{T})} \leqslant C\,||\widetilde{w}||_{H^1(\widetilde{T})} \leqslant C\,||\nabla_{\widetilde{s}}\widetilde{w}||_{\widetilde{T}}\,.$$

Concerning the left hand side of (2.36) we just need to use the inclusion $H^1(\partial\widetilde{T}) \subset H^{1/2}(\partial\widetilde{T})$ which gives,

$$\left|\left|\frac{\partial\widetilde{w}}{\partial n}_{|\widetilde{s}}\right|\right|_{\partial\widetilde{T}} \leqslant ||\widetilde{w}||_{H^{1/2}(\partial\widetilde{T})} \leqslant C\,||\nabla_{\widetilde{s}}\widetilde{w}||_{\widetilde{T}}\,.$$

Finally,

$$\left|\left|\frac{\partial\widetilde{w}}{\partial n}_{|\widetilde{s}}\right|\right|_{\partial\widetilde{T}} \leqslant C\,||\nabla_{\widetilde{s}}\widetilde{w}||_{\widetilde{T}}\,. \tag{2.37}$$

It remains to use the scaling propositions, Proposition 3 for the right hand side term and Proposition 4 for the left hand side. Then, we get for T a triangle of the mesh and E an edge which belongs to $\mathcal{E}_h^T \bigcap \mathcal{E}_h^I$,

$$h_E^{1/2}\left|\left|\frac{\partial w}{\partial n}\right|\right|_E \leqslant C\,||\nabla w||_T\,. \tag{2.38}$$

Now if we restrict (2.35) to an edge E of $\mathcal{E}_h^I$ which is shared by the two triangles T_1 and T_2 and detail the average in the left hand side we get,

$$\left|\left|h_E^{1/2}\left\{\frac{\partial(u-u_h^f)}{\partial n}\right\}\right|\right|_E = h_E^{1/2}\frac{1}{2}\left|\left|\frac{\partial(u-u_h^f)}{\partial n}_{|T_1} + \frac{\partial(u-u_h^f)}{\partial n}_{|T_2}\right|\right|_E$$

Using triangular inequality and (2.38) we get,

$$\begin{aligned}
h_E^{1/2}\frac{1}{2}\left|\left|\frac{\partial(u-u_h^f)}{\partial n}_{|T_1} + \frac{\partial u-u_h^f}{\partial n}_{|T_2}\right|\right|_E &\leqslant h_E^{1/2}\frac{1}{2}\left(\left|\left|\frac{\partial(u-u_h^f)}{\partial n}_{|T_1}\right|\right|_E + \left|\left|\frac{\partial u-u_h^f}{\partial n}_{|T_2}\right|\right|_E\right) \\
&\leqslant C\left(\left|\left|\nabla(u-u_h^f)\right|\right|_{T_1} + \left|\left|\nabla(u-u_h^f)\right|\right|_{T_2}\right).
\end{aligned}$$

So,

$$\left|\left|h_E^{1/2}\left\{\frac{\partial(u-u_h^f)}{\partial n}\right\}\right|\right|_E \leqslant C\left(\left|\left|\nabla(u-u_h^f)\right|\right|_{T_1} + \left|\left|\nabla(u-u_h^f)\right|\right|_{T_2}\right),$$

and taking the square and using its convexity gives,

$$\left\| h_E^{1/2} \left\{ \frac{\partial(u-u_h^f)}{\partial n} \right\} \right\|_E^2 \leqslant C \left(\left\| \nabla(u-u_h^f) \right\|_{T_1}^2 + \left\| \nabla(u-u_h^f) \right\|_{T_2}^2 \right),$$

and if we sum over the interior edges,

$$\sum_{E \in \mathcal{E}_h^I} \left(\left\| h_E^{1/2} \left\{ \frac{\partial(u-u_h^f)}{\partial n} \right\} \right\|_E^2 \right) \leqslant C \left\| \nabla(u-u_h^f) \right\|_\Omega^2 .$$

□

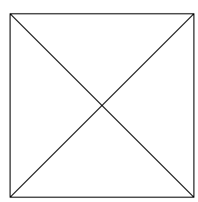
As we notice right below Theorem 6, this theorem is valid only under the saturation assumption. A natural question arise :

When this assumption it is valid ?

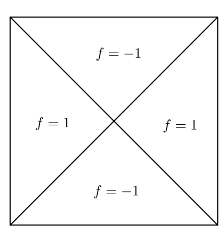
Unfortunately, we can find very simple cases which not verify this assumption. Here is a particulary simple example, given in [7]. We consider the Poisson equation,

$$-\Delta u = f,$$

with the Dirichlet boundary condition, on the domain Ω which is a square and on which we define a very simple mesh as follow,



Then, we take the data f piecewise constant as follow,



Moreover if we take $k = 1$, in other words if u_h is a continuous piecewise linear approximation of u and u_h^f a continuous piecewise quadratic approximation of u from the respective spaces V_h and V_h^f, we have,

$$\int_\Omega f\phi = 0, \quad \forall \phi \in V_h, V_h^f.$$

And this implies,

$$u_h = u_h^f = 0.$$

Then as a result, in this case the equation (2.34) of the saturation assumption gives,

$$||\nabla u|| \leqslant \alpha \, ||\nabla u|| \, ,$$

which is clearly false since $\alpha < 1$.

The work of W.Dorfler and R.H.Nochetto [7] suggests that the failure of this assumption is due to the too strong oscillations of the data f and to a too coarse mesh. In [7] they set the following result which link the saturation assumption to the oscillations of f at the patch level.

Theorem 7 ([7])**.** *There exists a constant* $0 < \mu < 1$ *solely depending on shape regularity of the mesh, but independant of* u *and* f*, such that if,*

$$\mathrm{osc}_{DN}(f) \leqslant \mu \, ||\nabla(u - u_h)||_\Omega$$

holds, then the saturation assumption (2.34) *is valid with* $\alpha := (1 - \mu^2)^{1/2}$.

In this theorem, the term of oscillations of f, namely $\mathrm{osc}_{DN}(f)$ represent the oscillations of f at the patch level. More precisely, for an interior node x of the mesh we denote

$$f_x := \mathrm{meas}(\eta_x)^{-1} \int_{\eta_x} f,$$

and

$$\mathrm{osc}_{DN}(f) := \left(\sum_{x \in \mathcal{N}_h^I} h_x \, ||f - f_x||_{\eta_x}^2 \right)^{1/2} .$$

Chapter 3

Equivalence between residual and BW estimators

In this chapter we will show a frame of the BW estimator with the residual estimator. The final goal is to frame the error e with the BW estimator, using the results on residual estimation and without any regularity hypothesis on the solution u, particulary without the saturation hypothesis in [3].

3.1 Upper-bound

Let us begin with the following theorem which gives an upper-bound for E_{res} :

Theorem 8. *There exists a positive constant $\overline{C}$ which depends only on the mesh regularity such that :*

$$E_{BW} \leqslant \overline{C}(E_{res}^2 + \mathrm{osc}^2(f))^{1/2}.$$

Proof. Let T be a triangle of the mesh $\mathcal{T}_h$. Let us take $v_h = \check{e} \in V_h^0(T)$ in the equation (2.23), introduce f_h an approximation of the data f which belongs to $V_h^g(T)$ and use the Cauchy-Schwarz inequality :

$$\begin{aligned} ||\nabla \check{e}||_T^2 &= \int_T (f_h + \Delta u_h)\check{e} \ + \frac{1}{2}\int_{\partial T} J_h \check{e} + \int_T (f - f_h)\check{e} \\ &\leqslant ||f_h + \Delta u_h||_T ||\check{e}||_T + \frac{1}{2}||J_h||_{\partial T}||\check{e}||_{\partial T} + ||f - f_h||_T ||\check{e}||_T. \end{aligned} \tag{3.1}$$

Then by Poincaré's inequality of Proposition1 (or by norm equivalence in finite dimension), we have for any $\widetilde{v}_h$ in $V_h^0(\widetilde{T})$,

$$||\widetilde{v}_h||_T \leqslant C\,||\nabla_{\tilde{s}}\widetilde{v}_h||\,,$$

with a constant C independant of h. Then, by Proposition 3, for any triangle T of the mesh if we take $\widetilde{v}_h = v_h \circ \mathcal{S}_T$,

$$||v_h||_T \leqslant Ch_T\,||\widetilde{v}_h||_{\widetilde{T}} \leqslant Ch_T\,||\nabla_{\tilde{s}}\widetilde{v}_h||_{\widetilde{T}} \leqslant Ch_T\,||\nabla_s v_h||_T\,.$$

So if we take $v_h = \check{e}$ we get,

$$||\check{e}||_T \leqslant C h_T ||\nabla \check{e}||_T. \tag{3.2}$$

By the Theorem 1 we have the existence of a constant C such that for any $\widetilde{v}_h$ in $V_h^0(\widetilde{T})$,

$$||\widetilde{v}_h||_{\partial \widetilde{T}} \leqslant C \, ||\widetilde{v}_h||_{H^1(\widetilde{T})} \, ,$$

and by Proposition 1,

$$||\widetilde{v}_h||_{\partial \widetilde{T}} \leqslant C \, ||\nabla_{\widetilde{s}} \widetilde{v}_h||_{\widetilde{T}} \, .$$

By Proposition 4 for the left hand side and Proposition 3 for the right hand side,

$$h_T^{-1/2} \, ||v_h||_{\partial T} \leqslant C \, ||\nabla_s v_h||_T \, ,$$

Finally if we take $v_h = \check{e}$,

$$||\check{e}||_{\partial T} \leqslant C h_T^{1/2} \, ||\nabla \check{e}||_T \, . \tag{3.3}$$

So, using (3.2) and (3.3) in (3.1) get,

$$||\nabla \check{e}||_T^2 \leqslant C\big(h_T ||f_h + \Delta u_h||_T ||\nabla \check{e}||_T + h_T ||f - f_h||_T ||\nabla \check{e}||_T + \sqrt{h_T} ||J_h||_{\partial T} ||\nabla \check{e}||_T\big),$$

If we divide by $||\nabla \check{e}||_T$ we obtain :

$$||\nabla \check{e}||_T \leqslant C \left(h_T ||f_h + \Delta u_h||_T + h_T ||f - f_h||_T + \sqrt{h_T} ||J_h||_{\partial T} \right) .$$

Now taking the square, using the convexity property and summing on the triangles of the mesh gives :

$$\sum_{T \in \mathcal{T}_h} ||\nabla \check{e}||_T^2 \leqslant C \left(\sum_{T \in \mathcal{T}_h} h_T^2 ||f_h + \Delta u_h||_T^2 + \sum_{T \in \mathcal{T}_h} h_T^2 ||f - f_h||_T^2 + \sum_{T \in \mathcal{T}_h} h_T ||J_h||_{\partial T}^2 \right) .$$

Finally, taking the square root, taking f_h which realize $\operatorname{osc}(f)$ and changing the last sum in a sum over the edges gives :

$$E_{\mathrm{BW}} \leqslant \overline{C} \left(E_{\mathrm{res}}^2 + \operatorname{osc}^2(f) \right)^{1/2} .$$

□

3.2 Lower-bound

Before setting the lower-bound we need to give some recalls about Legendre polynomials (see [6] Chap.4.7.8).

Definition 3 (Legendre polynomials)**.** *On the interval $I = [-1; 1]$ we define the n^{th} Legendre polynomial as follow,*

$$P_n(s) = \frac{1}{2^n n!} \left((s^2 - 1)^n \right)^{(n)} .$$

So defined, the family $\{P_n\}_{n\in\mathbb{N}}$ is an orthogonal basis of $L^2([-1;1])$, we also have for any n,

$$P_n(1) = 1,$$

and since, $P_n(-s) = (-1)^n P_n(s)$,

$$P_n(-1) = (-1)^n.$$

Furthermore the first Legendre polynomials are given by,

$$P_0(s) = 1, \quad P_1(s) = s, \quad P_2(s) = \frac{1}{2}(3s^2 - 1).$$

If E belongs to $\mathcal{E}_h$ let us now denote $(P_{E,n,x})_{n\in\mathbb{N}}$ the family of Legendre polynomials scaled to the edge E and if x and x' are the bounds of E we assume :

- $P_{E,n,x}(x) = 1$ and $P_{E,n,x}(x') = (-1)^n$,
- $\displaystyle\int_E P_{E,n,x} P_{E,l,x} = \delta_{n,l}$, for all n and l in $\mathbb{N}$,

where $\delta_{n,l}$ is the Kronecker symbol.

We will change a little bit the family $(P_{E,n,x})_{n\in\mathbb{N}}$ and we define the polynomial $L_{E,n,x}$ as :

$$L_{E,n,x} = \frac{P_{E,n,x} + P_{E,n+1,x}}{2},$$

such that :

- $L_{E,n,x}$ belongs to $\mathbb{P}_{n+1}(E)$,
- $L_{E,n,x}(x) = 1$ and $L_{E,n,x}(x') = 0$ for all $x' \in \mathcal{N}_h$,
- $\displaystyle\int_E L_{E,n,x} q = 0$ for all polynomial q in $\mathbb{P}_{n-1}(E)$.

We also need the following definition, in the case of $k = 1$,

Definition 4. *Let x be a mesh node in $\mathcal{N}_h$. We construct a function ψ_x associated to x such that :*

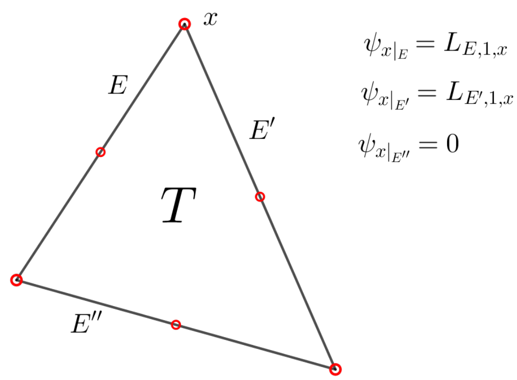
i) *ψ_x belongs to V_h^f,*

ii) *$\mathrm{supp}(\psi_x) = \eta_x$,*

iii) *$\psi_x = L_{E,1,x}$ for any edge E touching x.*

We also set $\widetilde{\psi}_0$ the function defined as above and associated to the node $(0,0)$ on the reference triangle $\widetilde{T}$.

The above function ψ_x is well defined. Indeed if $k = 1$, the degrees of freedom of the finite elements space V_h^f (which is a space of quadratic polynomials) are the vertices and the middle of each edge. To properly define ψ_x on a triangle T of η_x we just need to specify these values in each degree of freedom, and this is done as we can see on the next figure,



Since ψ_x is well defined on each triangle of η_x, so it is on η_x.

Now we give the main result, namely the lower-bound of E_{BW} by E_{res},

Theorem 9. *There exists a positive constant $\underline{C}$ only depending on the regularity of the mesh, such that :*

$$E_{res} \leqslant \underline{C}(E_{BW} + \operatorname{osc}(f)).$$

Proof. **We only give the proof for linear finite elements (i.e. k=1).**

We start by noticing that in this case since u_h is a piecewise linear polynomial, $\Delta u_h = 0$ and we have :

$$E_{\mathrm{res}} = \left(\sum_{T \in \mathcal{T}_h} h_T^2 ||f_h||_T^2 + \sum_{E \in \mathcal{E}_h} h_E \left|\left| \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right] \right|\right|_E^2 \right)^{1/2},$$

where f_h is a piecewise polynomial approximation of f which belongs to the space V_h^g. Then for $k = 1$, f_h is piecewise constant on each triangle. So on a triangle T of the mesh we take $f_h := f_T$ the average of f on T, in other terms :

$$f_T = \frac{1}{\operatorname{meas}(T)} \int_T f.$$

Since J_h is a polynomial function in $\mathbb{P}_0$ by definition of ψ_x we have

$$\sum_{E \in \mathcal{E}_h^I} \int_E J_h \psi_x = 0,$$

so if we take $v_h = \psi_x \in V_h^f$, (2.28) becomes :

$$\int_{\eta_x} \nabla \check{e} \cdot \nabla (\mathrm{Id} - \mathcal{I}) \psi_x = \int_{\eta_x} f \psi_x.$$

Introducing $f - f_h$ we get :

$$\int_{\eta_x} f_h \psi_x = \int_{\eta_x} \nabla \check{e} \cdot \nabla(\mathrm{Id} - \mathcal{I})\psi_x \;\; - \int_{\eta_x} (f - f_h)\psi_x. \tag{3.4}$$

Let us deal with the left hand side. First, since f_h is constant on each triangle we have for each triangle T :

$$\begin{aligned} ||f_h||_T^2 &= \int_T f_T^2 \\ &= f_T^2 \operatorname{meas}(T), \end{aligned}$$

so,

$$f_T = \frac{||f_h||_T}{\operatorname{meas}(T)^{1/2}}.$$

Then, setting $h_x = \max\limits_{T \in \eta_x}(h_T)$, using a similar argument than in 1. of Proposition 3 and using the local quasi-uniformity (1.4) gives :

$$\begin{aligned} \left| \int_{\eta_x} f_h \psi_x \right| &= \left| \sum_{T \in \eta_x} \int_T f_T \psi_x \right| \\ &= \left| \sum_{T \in \eta_x} f_T \int_T \psi_x \right| \\ &= \left| \sum_{T \in \eta_x} \frac{||f_h||_T}{\operatorname{meas}(T)^{1/2}} \int_T \psi_x \right| \\ &\geqslant \left| \sum_{T \in \eta_x} C \frac{||f_h||_T}{\operatorname{meas}(T)^{1/2}} h_T^2 \int_{\widetilde{T}} \widetilde{\psi}_0 \right| \\ &\geqslant \left| \sum_{T \in \eta_x} C \frac{||f_h||_T}{\operatorname{meas}(\eta_x)^{1/2}} h_T^2 \int_{\widetilde{T}} \widetilde{\psi}_0 \right| \\ &\geqslant \left| \sum_{T \in \eta_x} C \frac{||f_h||_T}{\operatorname{meas}(\eta_x)^{1/2}} \times \frac{h_x^2}{\delta_1^2} \int_{\widetilde{T}} \widetilde{\psi}_0 \right|. \end{aligned}$$

Yet, by (1.2) :

$$\operatorname{meas}(\eta_x)^{1/2} \leqslant h_x \operatorname{card}(\eta_x) \leqslant h_x C_0',$$

then

$$\left| \int_{\eta_x} f_h \psi_x \right| \geqslant \sum_{T \in \eta_x} C \, ||f_h||_T \, \frac{h_x}{\delta_1^2} \int_{\widetilde{T}} \widetilde{\psi}_0.$$

and finally

$$\left|\int_{\eta_x} f_h\psi_x\right| \geqslant C(\delta_1)h_x \sum_{T\in\eta_x} ||f_h||_T^2\,, \tag{3.5}$$

with $C(\delta_1) = \left|\dfrac{C}{C_0'\delta_1^2}\displaystyle\int_{\widetilde{T}}\widetilde{\psi}_0\right|$ is only depending on δ_1 of local quasi-uniformity of the mesh.

Now we estimate the right hand side of (3.4) using triangular inequality, Cauchy-Schwarz and since ψ_x belongs to V_h^f, Theorem 2 :

$$\begin{aligned}\left|\int_{\eta_x}\nabla\check{e}\cdot\nabla(\mathrm{Id}-\mathcal{I})\psi_x \;-\int_{\eta_x}(f-f_h)\psi_x\;\right| &\leqslant ||\nabla\check{e}||_{\eta_x}||\nabla(\mathrm{Id}-\mathcal{I})\psi_x||_{\eta_x}\\ &\qquad +||f-f_h||_{\eta_x}||\psi_x||_{\eta_x}\\ &\leqslant C||\psi_x||_{\eta_x}\left(h_x^{-1}||\nabla\check{e}||_{\eta_x}\right.\\ &\qquad \left.+||f-f_h||_{\eta_x}\right).\\ &\leqslant C\left|\left|\widetilde{\psi}_0\right|\right|_{\widetilde{T}}\left(h_x^{-1}||\nabla\check{e}||_{\eta_x}\right.\\ &\qquad \left.+||f-f_h||_{\eta_x}\right).\end{aligned} \tag{3.6}$$

If we gather (3.5) and (3.6) we obtain :

$$h_x||f_h||_{\eta_x} \leqslant C\big(||\nabla\check{e}||_{\eta_x} + h_x||f-f_h||_{\eta_x}\big),$$

where C depends on the interpolation operator $\mathcal{I}$ and on the regularity of the mesh but is independant of h.

Taking the square, using its convexity and summing on all nodes of the mesh get :

$$\sum_{x\in\mathcal{N}_h} h_x^2\,||f_h||_{\eta_x}^2 \leqslant C\sum_{x\in\mathcal{N}_h}\big(\,||\nabla\check{e}||_{\eta_x}^2 + h_x^2\,||f-f_h||_{\eta_x}^2\,\big). \tag{3.7}$$

Now notice that every triangle of the mesh is counted three times when we sum over all the nodes. Then if we change the sums over nodes into sums over triangles we obtain on one hand :

$$\sum_{x\in\mathcal{N}_h} h_x^2||f_h||_{\eta_x}^2 \geqslant 3\sum_{T\in\mathcal{T}_h} h_T^2||f_h||_T^2. \tag{3.8}$$

And on the other hand, using the local quasi-uniformity of the mesh (1.4) we have for any function v in $H_0^1(\Omega)$,

$$
\begin{aligned}
\sum_{x\in\mathcal{N}_h} h_x^2\,||\nabla v||_{\eta_x}^2 &= \sum_{x\in\mathcal{N}_h}\sum_{T\in\eta_x} h_x^2\,||\nabla v||_T^2 \\
&= \sum_{T\in\mathcal{T}_h}\sum_{x\ \text{s.t}\ T\in\eta_x} h_x^2\,||\nabla x||_T^2 \\
&\leqslant \sum_{T\in\mathcal{T}_h}\sum_{x\ \text{s.t.}\ T\in\eta_x} \delta_1^2 h_T^2\,||\nabla x||_T^2 \\
&\leqslant \sum_{T\in\mathcal{T}_h} 3\delta_1^2 h_T^2\,||\nabla x||_T^2\,.
\end{aligned}
$$

Applying this in the following sum we get,

$$
\begin{aligned}
C\left(\sum_{x\in\mathcal{N}_h} ||\nabla\check{e}||_{\eta_x}^2 + \sum_{x\in\mathcal{N}_h} h_x^2||f-f_h||_{\eta_x}^2\right) &\leqslant 3C\left(\sum_{T\in\mathcal{T}_h} ||\nabla\check{e}||_T^2 + \delta_1^2\sum_{T\in\mathcal{T}_h} h_T^2||f-f_h||_T^2\right) \\
&\leqslant 3C\left(\sum_{T\in\mathcal{T}_h}(||\nabla\check{e}||_T^2) + \delta_1^2\operatorname{osc}^2(f)\right).
\end{aligned} \tag{3.9}
$$

Then combining (3.7), (3.8) and (3.9) we finally get :

$$
\sum_{T\in\mathcal{T}_h} h_T^2||f_h||_T^2 \leqslant C\left(\sum_{T\in\mathcal{T}_h}(||\nabla\check{e}||_T^2) + \delta_1^2\operatorname{osc}^2(f)\right). \tag{3.10}
$$

with a constant C only depending on $\mathcal{I}$ and on δ_1 of the local quasi-uniformity.

It remains now to bound the terms in E_{res} containing the jumps J_h. This proof can be done in the same way for any $k\in\mathbb{N}$ but is based on the previous step. Consider also b_E the usual bubble function defined in Definition 2. Let us set the following application for an integer l,

$$
\begin{aligned}
\mathbb{P}_l(\breve{E}) &\longrightarrow \mathbb{P}_l(E) \\
p_h &\longmapsto p_{h_{|E}},
\end{aligned}
$$

this application maps a polynomial defined on $\breve{E}$ to its restriction to E. Since this application is a surjection, we can extend any polynomial g_h in $\mathbb{P}_l(E)$ to a polynomial of $\mathbb{P}_l(\breve{E})$. So lets extend J_h from E to $\breve{E}$ by a polynomial still denoted by J_h. Since J_h belongs to $\mathbb{P}_{k-1}(\breve{E})$ and b_E belongs to $\mathbb{P}_2(\breve{E})$ we have that J_hb_E belongs to $V_h^f(\breve{E})$. Now if we set $v_h = J_hb_E$ in (2.25) we get :

$$
\int_E J_h^2 b_E \;=\int_{\breve{E}}\nabla\check{e}\cdot\nabla(\mathrm{Id}-\mathcal{I})J_hb_E\;-\int_{\breve{E}} rJ_hb_E.
$$

By Cauchy-Schwarz and Theorem 2 :

$$
\int_E J_h^2 b_E\;\leqslant\left(C_L h_{\breve{E}}^{-1}||\nabla\check{e}||_{\breve{E}} + ||r||_{\breve{E}}\right)||J_hb_E||_{\breve{E}}.
$$

By shape regularity of the mesh, it exists a constant C only depending on δ_0 such that :

$$h_{\breve{E}}^{-1} \leqslant C h_E^{-1},$$

then :

$$\int_E J_h^2 b_E \leqslant \left(C_L C h_E^{-1} ||\nabla \check{e}||_{\breve{E}} + ||r||_{\breve{E}}\right) ||J_h b_E||_{\breve{E}}.$$

Moreover, by Proposition 7 we have

$$\int_E J_h^2 \leqslant C \int_E J_h^2 b_E,$$

then :

$$\int_E J_h^2 \leqslant C \left(h_E^{-1} ||\nabla \check{e}||_{\breve{E}} + ||r||_{\breve{E}}\right) ||J_h b_E||_{\breve{E}}.$$

Also by Proposition 7 we get

$$||J_h b_E||_{\breve{E}} \leqslant C h_E^{1/2} ||J_h||_E.$$

So, using this last inequality and multiplying by $h_E^{1/2}$ gives :

$$h_E^{1/2} \int_E J_h^2 \leqslant C \left(||\nabla \check{e}||_{\breve{E}} + h_E ||r||_{\breve{E}}\right) ||J_h||_E.$$

Dividing by $||J_h||_E$, taking the square and using its convexity :

$$h_E ||J_h||_E^2 \leqslant C \left(||\nabla \check{e}||_{\breve{E}}^2 + h_E^2 ||r||_{\breve{E}}^2\right).$$

And if we sum over all edges of the mesh and changes the sums of the right hand side into sums on triangles we get :

$$\sum_{E \in \mathcal{E}_h} h_E ||J_h||_E^2 \leqslant 3C \left(\sum_{T \in \mathcal{T}_h} ||\nabla \check{e}||_T^2 + \sum_{T \in \mathcal{T}_h} h_T^2 ||r||_T^2\right). \tag{3.11}$$

Now by choosing f_h such that $\operatorname{osc}(f)$ is realised, by triangular inequality and by convexity of the square we have :

$$\begin{aligned}
\sum_{T \in \mathcal{T}_h} h_T^2 ||f + \Delta u_h||_T^2 &\leqslant 2 \left(\sum_{T \in \mathcal{T}_h} h_T^2 ||f_h + \Delta u_h||_T^2 + \sum_{T \in \mathcal{T}_h} h_T^2 ||f - f_h||_T^2\right) \\
&\leqslant 2 \left(\sum_{T \in \mathcal{T}_h} h_T^2 ||f_h + \Delta u_h||_T^2 + \operatorname{osc}(f)^2\right).
\end{aligned} \tag{3.12}$$

And for bound the term $\sum_{T \in \mathcal{T}_h} h_T^2 ||f_h + \Delta u_h||_T^2$ we can use (3.10) (for the case $k = 1$) and we get :

$$\sum_{T \in \mathcal{T}_h} h_T^2 ||f_h + \Delta u_h||_T^2 \leqslant C \left(\sum_{T \in \mathcal{T}_h} ||\nabla \check{e}||_T^2 + \operatorname{osc}(f)^2\right). \tag{3.13}$$

Then combining (3.11), (3.12) and (3.13) we obtain :

$$\sum_{E \in \mathcal{E}_h} h_E ||J_h||_E^2 \leqslant C \left(\sum_{T \in \mathcal{T}_h} ||\nabla \check{e}||_T^2 + \mathrm{osc}(f)^2 \right), \tag{3.14}$$

where C depends on the interpolant $\mathcal{I}$ and on the mesh regularity but does not depend on h.

Finally, combining and (3.10), taking the square root and using convexity of the square gives the result :

$$\left(\sum_{T \in \mathcal{T}_h} h_T^2 ||r_h||_T^2 + \sum_{E \in \mathcal{E}_h} h_E ||J_h||_E^2 \right)^{1/2} \leqslant C \left(||\nabla \check{e}||_\Omega^2 + \mathrm{osc}^2(f) \right)^{1/2},$$

with a constant C which depends only on the mesh regularity and the interpolant $\mathcal{I}$.

□

Chapter 4

Conclusion and opening

The main idea of this work was to maintain a frame as general as possible concerning the regularity of the solution u to show that the a posteriori error estimator of Bank-Weiser has good properties of convergence even when the problem concerned admits a singuliar solution.

The first step was to do whithout the saturation assumption which exclude this kind of problems. The next step is naturally to generalize the result of the lower-bound to finite elements spaces of higher order polynomials.

In parallel, a numerical study of the BW estimator on test problems which admit singuliar solutions (like on a "L" domain or on a slit square) could be give an idea of the efficiency and the behavior of this estimator.

Another way to study the convergence properties of BW estimator would be through its asymptotic exactness. The asymptotic exactness of BW estimator is not guaranteed according to the present constants which appear in the estimations of Theorem 6. However, in their work [8], R.Durán and R.Rodriguez have already proved that for problems which admit reguliar solutions (H^3) and for particulary reguliar meshes (parallel meshes) the BW estimator is asymptotically exact.

Bibliography

[1] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] Mark Ainsworth and J. Tinsley Oden. *A posteriori error estimation in finite element analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2000.

[3] R. E. Bank and A. Weiser. Some a posteriori error estimators for elliptic partial differential equations. *Math. Comp.*, 44(170):283–301, 1985.

[4] C. Bernardi and V. Girault. A local regularization operator for triangular and quadrilateral finite elements. *SIAM J. Numer. Anal.*, 35(5):1893–1916, 1998.

[5] Haïm Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[6] Srishti D. Chatterji. *Cours d'analyse. 3*. Mathématiques. [Mathematics]. Presses Polytechniques et Universitaires Romandes, Lausanne, 1998. Équations différentielles ordinaires et aux dérivées partielles. [Ordinary and partial differential equations].

[7] Willy Dörfler and Ricardo H. Nochetto. Small data oscillation implies the saturation assumption. *Numer. Math.*, 91(1):1–12, 2002.

[8] Ricardo Durán and Rodolfo Rodrí guez. On the asymptotic exactness of Bank-Weiser's estimator. *Numer. Math.*, 62(3):297–303, 1992.

[9] Alexandre Ern and Jean-Luc Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[10] Ricardo H. Nochetto. Removing the saturation assumption in a posteriori error analysis. *Istit. Lombardo Accad. Sci. Lett. Rend. A*, 127(1):67–82 (1994), 1993.