A posteriori error estimation in the FEniCSx finite element software and application to the fractional Laplacian.

Raphaël Bulle

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Background

2010-2013 **Bachelor in Mathematics**

at Université de Bourgogne Franche-Comté (FR).

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2014 **CAPES** (competitive exam)

of Mathematics.

2010-2013 **Bachelor in Mathematics**

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2016 **Agrégation** (competitive exam)

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Background

2015-2017 **Master in Advanced Mathematics**

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2014 **CAPES** (competitive exam)

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2010-2013 **Bachelor in Mathematics**

at Université de Bourgogne Franche-Comté (FR).

Background

2017-2022 **PhD student in Computational Engineering and Applied Mathematics**
at University of Luxembourg and Université de Bourgogne Franche-Comté

Supervision: S. P. A. Bordas, F. Chouly, J. S. Hale and A. Lozinski.

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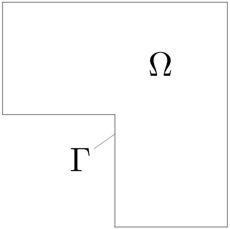

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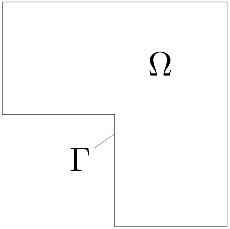

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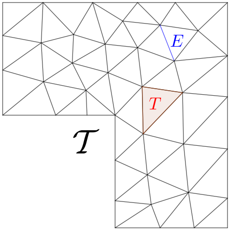

Lagrange finite element discretization of order k, find u_k in V^k such that

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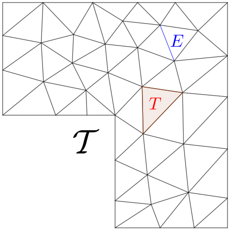

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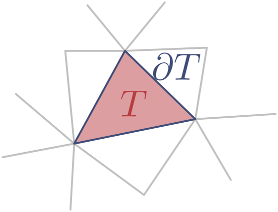
Goal: estimate $\eta_{\mathrm{err}} = |||u_k - u|||_\Omega$ i.e. find a computable quantity η_{bw} such that $\eta_{\mathrm{bw}} \approx \eta_{\mathrm{err}}$.

The Bank–Weiser estimator

Definition



On a cell T, the Bank–Weiser problem is given by: find e_T^{bw} in $\boldsymbol{V_T^{\mathrm{bw}}}$ such that

$$\int_T e_T^{\mathrm{bw}} v_T^{\mathrm{bw}} + a \int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T^{\mathrm{bw}} = \int_T r_T v_T^{\mathrm{bw}} + \sum_{E \in \partial T} \frac{1}{2} \int_E J_E v_T^{\mathrm{bw}} \quad \forall v_T^{\mathrm{bw}} \in V_T^{\mathrm{bw}}.$$

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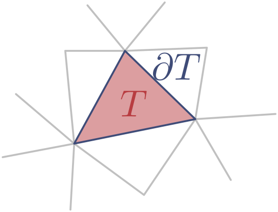

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The Bank–Weiser estimator is defined as

$$\eta_{\mathrm{bw}}^2 := \sum_{T \in \mathcal{T}} \eta_{\mathrm{bw},T}^2, \quad \eta_{\mathrm{bw},T} := |||e_T^{\mathrm{bw}}|||_T.$$

The Bank–Weiser estimator

Definition

What is V_T^{bw} ?

Let $V_T^- \subsetneq V_T^+$ be two finite element spaces and

$$\mathcal{L}_T : V_T^+ \longrightarrow V_T^-,$$

be the local Lagrange interpolation operator,

$$V_T^{\mathrm{bw}} := \ker(\mathcal{L}_T) = \left\{ v_T^+ \in V_T^+,\ \mathcal{L}_T(v_T^+) = 0 \right\}.$$

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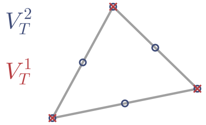

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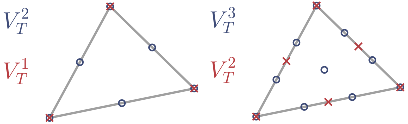

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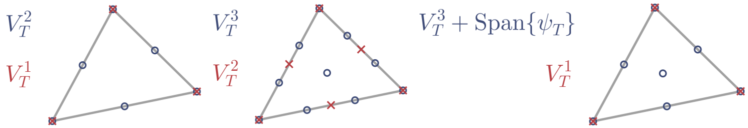

The Bank–Weiser estimator

Implementation

We need to compute the matrix A_T^{bw} and vector b_T^{bw} from

$$\int_T e_T^{\mathrm{bw}} v_T^{\mathrm{bw}} + a \int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T^{\mathrm{bw}} = \int_T r_T v_T^{\mathrm{bw}} + \sum_{E \in \partial T} \frac{1}{2} \int_E J_E v_T^{\mathrm{bw}} \quad \forall v_T^{\mathrm{bw}} \in V_T^{\mathrm{bw}}.$$

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Problem: the space V_T^{bw} is not provided by DOLFIN/x.

The Bank–Weiser estimator

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Problem: the space V_T^{bw} is not provided by DOLFIN/x.
Idea: we rely on the matrix A_T^+ and vector b_T^+ from

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since V_T^+ is provided by DOLFIN/x

The Bank–Weiser estimator

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We need to compute the matrix A_T^{bw} and vector b_T^{bw} from

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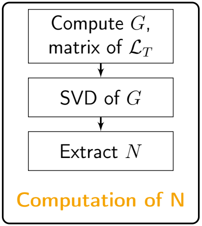
since V_T^+ is provided by DOLFIN/x and we look for a matrix N such that:

$$A_T^{\mathrm{bw}} = N^{\mathsf{t}} A_T^+ N, \quad \text{and} \quad b_T^{\mathrm{bw}} = N^{\mathsf{t}} b_T^+.$$

The Bank–Weiser estimator

Implementation



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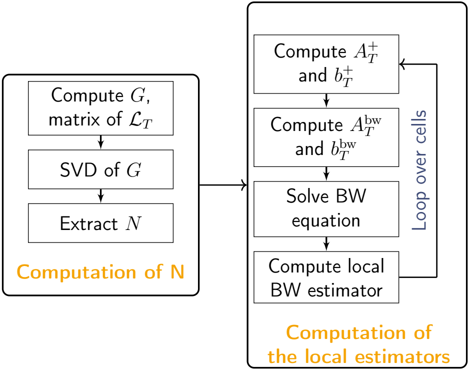

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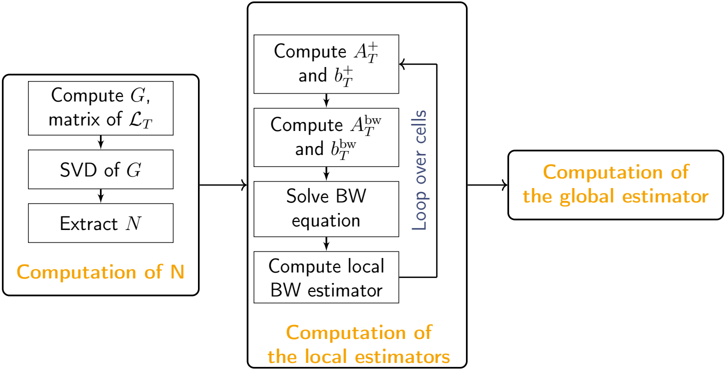

The Bank–Weiser estimator

Pros and cons

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Local efficiency
$\eta_{\mathrm{bw},T} \leqslant C\eta_{\mathrm{err},T} + \mathrm{h.o.t.}_{|_T}$
[Bank and Weiser, 1985, Nochetto, 1993, Verfürth, 1994].

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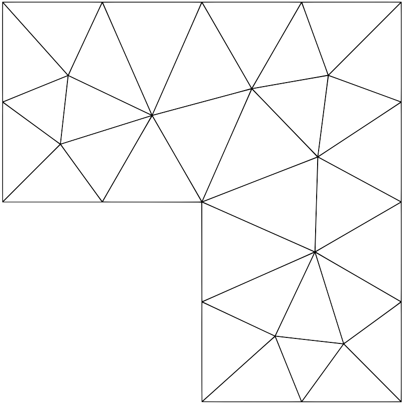
[Ainsworth, 1994].

No convergence proof when used for adaptive mesh refinement
[Carstensen et al., 2014].

The Bank–Weiser estimator

Numerical results

Adaptive finite elements for a Poisson problem:
$-\Delta u = 0$ in Ω, $\quad u = u_D$ on Γ. Linear finite elements.



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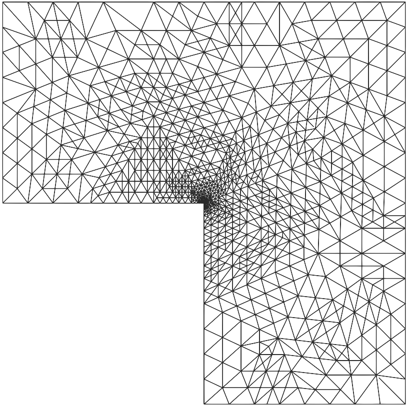

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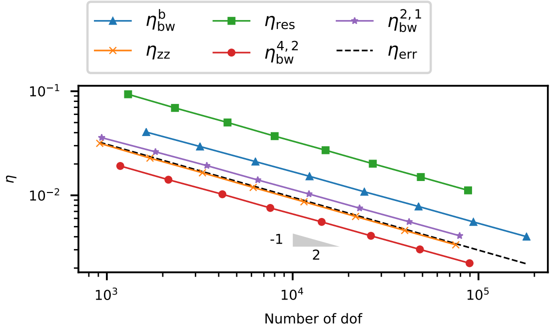
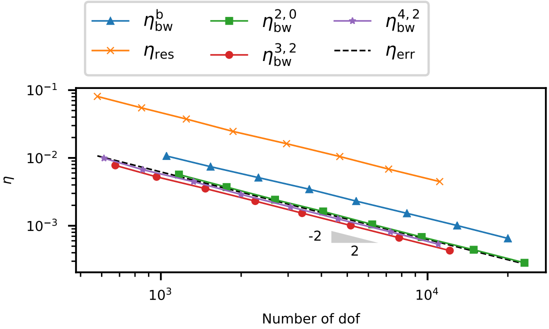


Notation	V_T^+	V_T^-
$\eta_{\mathrm{bw}}^{k_+,k_-}$	$V_T^{k_+}$	$V_T^{k_-}$
η_{bw}^{b}	$V_T^2 + \mathrm{bubble}$	V_T^1

Adaptive finite elements for a Poisson problem:
$-\Delta u = 0$ in Ω, $u = u_D$ on Γ. Quadratic finite elements.



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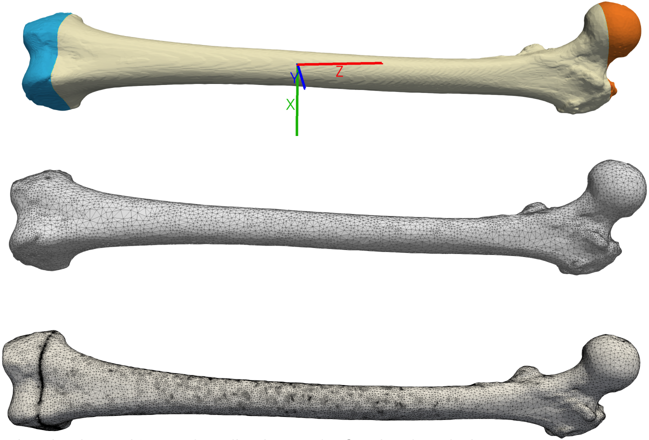
The Bank–Weiser estimator

Numerical results

GO AFEM for a linear elasticity problem:

we used a technique from [Khan et al., 2019] and [Becker et al., 2011] to compute the estimators. The goal functional is defined by $J(\mathbf{u}, p) := \int_\Gamma \mathbf{u} \cdot \mathbf{n} c$, where c is a Gaussian weight centered on the middle of the bone.



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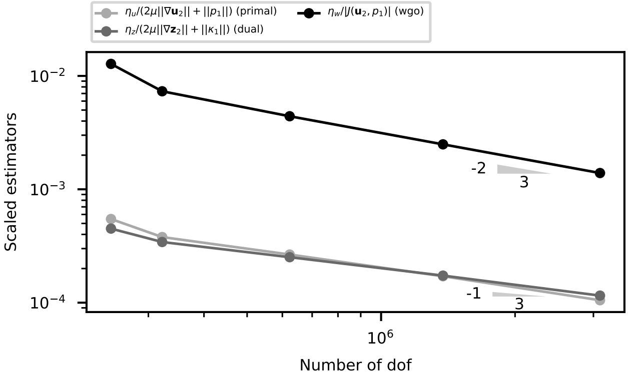
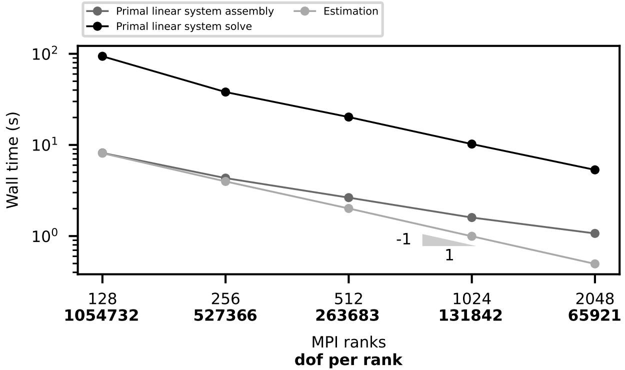


The Bank–Weiser estimator

Numerical results

Timescale study:
strong scaling study on the Uni Lu cluster [Varrette et al., 2014].
$-\Delta u = f$ on $[0, 1]^3$, $u = 0$ on Γ. $\mathcal{P}_2$ Lagrange elements.
The Bank–Weiser estimator is $\eta_{\mathrm{bw}}^{3,2}$.
The problem size is fixed around 135 million dof.



The spectral fractional Laplacian

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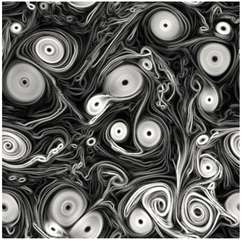
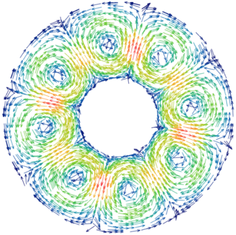


[Bonito and Nazarov, 2021]



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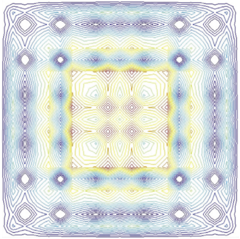
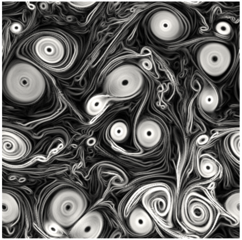


[Sumelka, 2015]

Fractional models are more and more popular and are used in a wide range of fields such as statistics, hydrogeology, finance, physics...

The spectral fractional Laplacian



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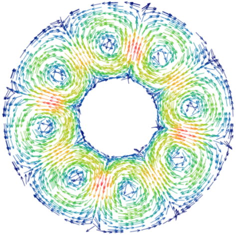

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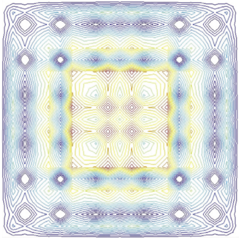

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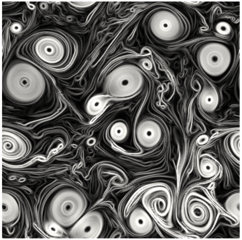
- **Main advantage:** they are nonlocal.

The spectral fractional Laplacian



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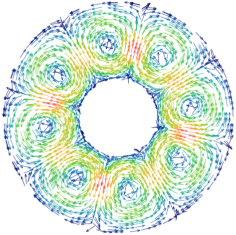

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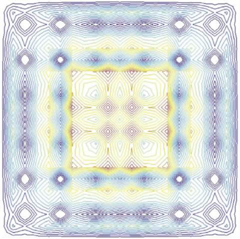

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Fractional models are more and more popular and are used in a wide range of fields such as statistics, hydrogeology, finance, physics...

- **Main advantage:** they are nonlocal.
- **Main drawback:** they are nonlocal.

The spectral fractional Laplacian

Problem setting

Let $\Omega \subset \mathbb{R}^d$, $s \in (0,1)$ and $f \in L^2(\Omega)$.

$$(-\Delta)^s u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma.$$

The spectral fractional Laplacian

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$$(-\Delta)^s u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma.$$

The solution u is defined by

$$u := \sum_{i=1}^{+\infty} \lambda_i^{-s} \left(f, \psi_i\right)_{L^2},$$

where $\{\psi_i, \lambda_i\}_{i=1}^{+\infty} \subset L^2(\Omega) \times \mathbb{R}^+$ is the spectrum of $-\Delta$.

The spectral fractional Laplacian

Problem setting

The natural space associated with this problem is

$$\mathbb{H}^s(\Omega) := \left\{ v \in L^2(\Omega),\ \sum_{i=1}^{+\infty} \lambda_i^s \, (v, \psi_i)_{L^2}^2 < +\infty \right\},$$

of norm $\|v\|_{\mathbb{H}^s}^2 := \sum_{i=1}^{+\infty} \lambda_i^s \, (v, \psi_i)_{L^2}^2$.

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Especially, $\mathbb{H}^0(\Omega) = L^2(\Omega)$ and $\mathbb{H}^1(\Omega) = H_0^1(\Omega)$.

The spectral fractional Laplacian

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If $f \in L^2(\Omega)$, then $u \in \mathbb{H}^{2s}(\Omega)$ and

$$\|u\|_{\mathbb{H}^{2s}}^2 = \|f\|_{L^2}^2.$$

The spectral fractional Laplacian

Discretization

$$(-\Delta)^s u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma.$$

How to solve this equation numerically ?

The spectral fractional Laplacian

Discretization

$$(-\Delta)^s u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma.$$

How to solve this equation numerically ?

$$u := (-\Delta)^{-s} f = \sum_{i=1}^{+\infty} \lambda_i^{-s} \left(f, \psi_i\right)_{L^2},$$

we use a rational approximation

$$\lambda^{-s} \simeq \mathcal{Q}_s^N(\lambda) := C_s(N) \sum_{l=1}^{N} a_l (1 + b_l \lambda)^{-1}, \qquad \forall \lambda \in [\lambda_1, +\infty),$$

where $(a_l)_l$ and $(b_l)_l$ are positive coefficients and $C_s(N)$ is independent of λ.

The spectral fractional Laplacian

Discretization

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The spectral fractional Laplacian

Discretization

$$u = (-\Delta)^{-s} f = \sum_{i=1}^{+\infty} \lambda_i^{-s} \, (f, \psi_i)_{L^2}$$

$$\simeq \sum_{i=1}^{+\infty} \mathcal{Q}_s^N(\lambda_i) \, (f, \psi_i)_{L^2}$$

The spectral fractional Laplacian

Discretization

$$
\begin{aligned}
u = (-\Delta)^{-s} f &= \sum_{i=1}^{+\infty} \lambda_i^{-s} \left(f, \psi_i\right)_{L^2} \\
&\simeq \sum_{i=1}^{+\infty} \mathcal{Q}_s^N(\lambda_i) \left(f, \psi_i\right)_{L^2} \\
&\simeq \sum_{i=1}^{+\infty} C_s(N) \sum_{l=1}^{N} a_l (1 + b_l \lambda_i)^{-1} \left(f, \psi_i\right)_{L^2}
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The spectral fractional Laplacian

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&\simeq C_s(N) \sum_{l=1}^{N} a_l \sum_{i=1}^{+\infty} (1 + b_l \lambda_i)^{-1} \left(f, \psi_i\right)_{L^2} \\
&\simeq C_s(N) \sum_{l=1}^{N} a_l (\mathrm{Id} - b_l \Delta)^{-1} f = C_s(N) \sum_{l=1}^{N} a_l u_l .
\end{aligned}
$$

The spectral fractional Laplacian

Discretization

$$(-\Delta)^s u = f, \quad \text{in } \Omega,$$
$$u = 0, \quad \text{on } \Gamma.$$

$\longrightarrow$

For $l = 1, \cdots, N$,

$$u_l - b_l \Delta u_l = f, \quad \text{in } \Omega, \quad (1)$$
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The spectral fractional Laplacian

Discretization

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We denote $u \simeq u_{\mathcal{Q}_s^N} := C_s(N) \sum_{l=1}^{N} a_l u_l$.

However, $u_{\mathcal{Q}_s^N}$ **is not a discrete function.**

The spectral fractional Laplacian

Discretization

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However, $u_{\mathcal{Q}_s^N}$ **is not a discrete function.** To get a full discretization, we use a FE method. We reformulate (1) and (2) in weak form

$$\int_\Omega u_l v + b_l \int_\Omega \nabla u_l \cdot \nabla v = \int_\Omega f v, \qquad \forall v \in H_0^1(\Omega), \ \forall l \in [\![1, N]\!],$$

The spectral fractional Laplacian

Discretization

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and write its FE discretization

$$\int_\Omega u_{l,k} v_k + b_l \int_\Omega \nabla u_{l,k} \cdot \nabla v_k = \int_\Omega f v_k, \qquad \forall v_k \in V^k,\ \forall l \in [\![1, N]\!].$$

The spectral fractional Laplacian

Discretization

Solving these classical FE problems we finally get a fully discrete approximation of u

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The spectral fractional Laplacian

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Advantages of this method:

- it is easily parallelizable,
- it involves "standard" FE machinery,
- it is well–suited to three–dimensional problems.

The spectral fractional Laplacian

Error estimation

How can we bound the discretization error ?

$$\|u - u^k_{\mathcal{Q}^N_s}\| \leqslant \|u - u_{\mathcal{Q}^N_s}\| + \|u_{\mathcal{Q}^N_s} - u^k_{\mathcal{Q}^N_s}\|.$$

where $\|\cdot\| = \|\cdot\|_{L^2}$, or $\|\cdot\|_{\mathbb{H}^s}$.

The spectral fractional Laplacian

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The spectral fractional Laplacian

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The spectral fractional Laplacian

Error estimation

Quantification of the rational approximation error $\|u - u_{\mathcal{Q}_s^N}\|$.

The spectral fractional Laplacian

Error estimation

Quantification of the rational approximation error $\|u - u_{\mathcal{Q}_s^N}\|$.

If there exists $\varepsilon_s(N) \underset{N\to+\infty}{\longrightarrow} 0$ such that

$$\left|\lambda^{-s} - \mathcal{Q}_s^N(\lambda)\right| \leqslant \varepsilon_s(N), \qquad \forall \lambda \in [\lambda_1, +\infty),$$

then, [Bonito and Pasciak, 2015]

$$\|u - u_{\mathcal{Q}_s^N}\|_{L^2} \leqslant \varepsilon_s(N)\|f\|_{L^2}.$$

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$$\|u - u_{\mathcal{Q}_s^N}\|_{L^2} \leqslant \varepsilon_s(N)\|f\|_{L^2}.$$

Moreover, if $f \in \mathbb{H}^s(\Omega)$ then [Bonito and Pasciak, 2016]

$$\|u - u_{\mathcal{Q}_s^N}\|_{\mathbb{H}^s} \leqslant \varepsilon_s(N)\|f\|_{\mathbb{H}^s}.$$

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$$\|u - u_{\mathcal{Q}_s^N}\|_{\mathbb{H}^s} \leqslant \varepsilon_s(N)\|f\|_{\mathbb{H}^s}.$$

In particular, there exists an approximation $\mathcal{Q}_s^N$ such that $\varepsilon_s(N)$ is fully computable and [Bonito and Pasciak, 2015]

$$\varepsilon_s(N) = \underset{N\to+\infty}{\mathcal{O}} \left(\mathrm{e}^{-\left(\pi^2/2\sqrt{2}\right)\sqrt{N}}\right).$$

The spectral fractional Laplacian

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Quantification of the rational approximation error $\|u - u_{\mathcal{Q}_s^N}\|$.

Conjecture: $\|u - u_{\mathcal{Q}_s^N}\|_{\mathbb{H}^s} \leqslant \varepsilon_s(N)\|f\|_{L^2}$.

The spectral fractional Laplacian

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Conjecture: $\|u - u_{\mathcal{Q}_s^N}\|_{\mathbb{H}^s} \leqslant \varepsilon_s(N)\|f\|_{L^2}$.

What we can prove [Bulle, 2022]:

$$\|u - u_{\mathcal{Q}_s^N}\|_{\mathbb{H}^s} \leqslant \widetilde{\varepsilon}_s(N)\|f\|_{L^2},$$

where $\widetilde{\varepsilon}_s(N) \underset{N\to+\infty}{\longrightarrow} 0$ with a possibly slower convergence rate than ε_s.

The spectral fractional Laplacian

Error estimation

Quantification of the finite element error $\|u_{\mathcal{Q}_s^N} - u^k_{\mathcal{Q}_s^N}\|$.

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- A posteriori error estimate for $\|\cdot\|_{\mathbb{H}^s}$ is an ongoing work.

The spectral fractional Laplacian

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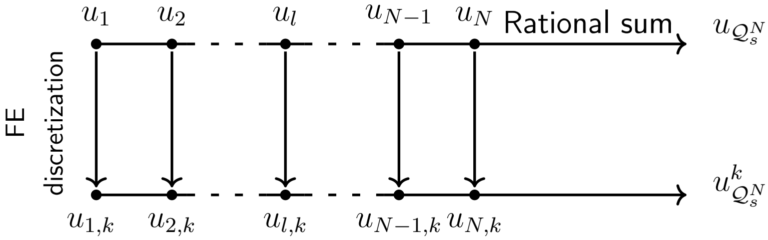
Quantification of the finite element error $\|u_{\mathcal{Q}_s^N} - u_{\mathcal{Q}_s^N}^k\|$.



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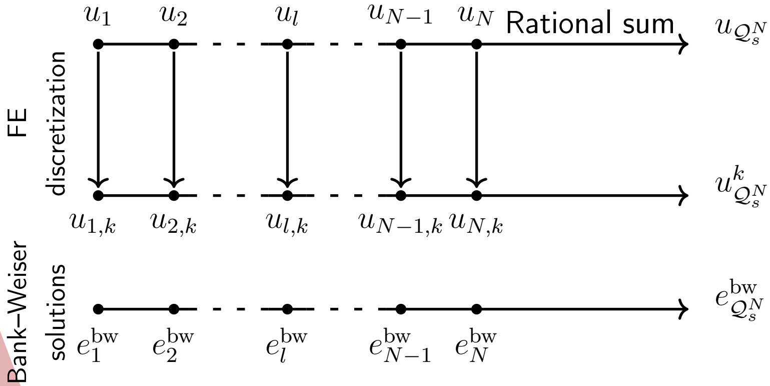

The spectral fractional Laplacian

Error estimation

Quantification of the finite element error $\|u_{\mathcal{Q}_s^N} - u^k_{\mathcal{Q}_s^N}\|$.

For each cell $T \in \mathcal{T}_h$ and each parametric problem $l \in [\![1, N]\!]$, we solve the Bank–Weiser equation to estimate the difference

$$u_{l|_T} - u_{l,k|_T} \simeq e^{\mathrm{bw}}_{l,T}.$$

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Then,

$$\left(u_{\mathcal{Q}_s^N} - u^k_{\mathcal{Q}_s^N}\right)_{|_T} = C_s(N) \sum_{l=1}^{N} a_l (u_l - u_{l,k}) \simeq C_s(N) \sum_{l=1}^{N} a_l e^{\mathrm{bw}}_{l,T} =: e^{\mathrm{bw}}_{\mathcal{Q}_s^N, T},$$

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and finally, we expect that:

$$\|u_{\mathcal{Q}_s^N} - u^k_{\mathcal{Q}_s^N}\|^2_{L^2} \simeq \|e^{\mathrm{bw}}_{\mathcal{Q}_s^N}\|^2_{L^2} = \sum_{T \in \mathcal{T}} \|e^{\mathrm{bw}}_{\mathcal{Q}_s^N, T}\|^2_{L^2(T)}.$$

The spectral fractional Laplacian

Adaptive mesh refinement

Fractional Laplacian problems show a particular sensibility to **boundary layers effect**. Thus, even for smooth data, the mesh might need to be adaptively refined near the boundary Γ [Banjai et al., 2019].

The spectral fractional Laplacian

Adaptive mesh refinement

Fractional Laplacian problems show a particular sensibility to **boundary layers effect**. Thus, even for smooth data, the mesh might need to be adaptively refined near the boundary Γ [Banjai et al., 2019].
We can use the Bank–Weiser error estimator to steer an adaptive refinement algorithm.

The spectral fractional Laplacian

Adaptive mesh refinement

Choose a tolerance $\delta > 0$, an initial mesh $\mathcal{T}_{n=0}$ and N such that $\varepsilon_s(N)\|f\|_{L^2} \ll \delta$
Generate the rational approximation $\mathcal{Q}_s^N$ coefficients
Initialize the estimator $\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N} = \delta + 1$

The spectral fractional Laplacian

Adaptive mesh refinement

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Generate the rational approximation $\mathcal{Q}_s^N$ coefficients
Initialize the estimator $\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N} = \delta + 1$

While $\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N} > \delta$:

- Initialize the solution $u_{\mathcal{Q}_s^N,k} = 0$
- Initialize the local Bank–Weiser solutions $\{e^{\mathrm{bw}}_{\mathcal{Q}_s^N,T} = 0\}_T$

The spectral fractional Laplacian

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Initialize the solution $u_{\mathcal{Q}_s^N,k} = 0$
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For each parametric problem $l \in [\![1, N]\!]$:

Solve parametric problem on $\mathcal{T}_n$ to obtain $u_{l,k}$
Add $u_{\mathcal{Q}_s^N,k} + C_s(N)a_l u_{l,k}$

The spectral fractional Laplacian

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While $\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N} > \delta$:

- Initialize the solution $u_{\mathcal{Q}_s^N,k} = 0$
  Initialize the local Bank–Weiser solutions $\{e^{\mathrm{bw}}_{\mathcal{Q}_s^N,T} = 0\}_T$

  For each parametric problem $l \in [\![1, N]\!]$:

  - Solve parametric problem on $\mathcal{T}_n$ to obtain $u_{l,k}$
    Add $u_{\mathcal{Q}_s^N,k} + C_s(N) a_l u_{l,k}$

    For each cell T of $\mathcal{T}_n$:

    - Solve BW local parametric problem on T to obtain $e^{\mathrm{bw}}_{l,T}$
      Add $e^{\mathrm{bw}}_{\mathcal{Q}_s^N,T} + C_s(N) a_l e^{\mathrm{bw}}_{l,T}$

The spectral fractional Laplacian

Adaptive mesh refinement

Choose a tolerance $\delta > 0$, an initial mesh $\mathcal{T}_{n=0}$ and N such that $\varepsilon_s(N)\|f\|_{L^2} \ll \delta$
Generate the rational approximation $\mathcal{Q}_s^N$ coefficients
Initialize the estimator $\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N} = \delta + 1$

While $\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N} > \delta$:

- Initialize the solution $u_{\mathcal{Q}_s^N,k} = 0$
- Initialize the local Bank–Weiser solutions $\{e^{\mathrm{bw}}_{\mathcal{Q}_s^N,T} = 0\}_T$
- **For** each parametric problem $l \in [\![1, N]\!]$:
  - Solve parametric problem on $\mathcal{T}_n$ to obtain $u_{l,k}$
  - Add $u_{\mathcal{Q}_s^N,k} + C_s(N) a_l u_{l,k}$
  - **For** each cell T of $\mathcal{T}_n$:
    - Solve BW local parametric problem on T to obtain $e^{\mathrm{bw}}_{l,T}$
    - Add $e^{\mathrm{bw}}_{\mathcal{Q}_s^N,T} + C_s(N) a_l e^{\mathrm{bw}}_{l,T}$
- Compute $\{\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N,T} = \|e^{\mathrm{bw}}_{\mathcal{Q}_s^N,T}\|_{L^2(T)}\}_T$
- Take the square root of the sum of $\{{\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N,T}}^2\}_T$ to obtain $\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N}$

The spectral fractional Laplacian

Adaptive mesh refinement

Choose a tolerance $\delta > 0$, an initial mesh $\mathcal{T}_{n=0}$ and N such that $\varepsilon_s(N)\|f\|_{L^2} \ll \delta$
Generate the rational approximation $\mathcal{Q}_s^N$ coefficients
Initialize the estimator $\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N} = \delta + 1$

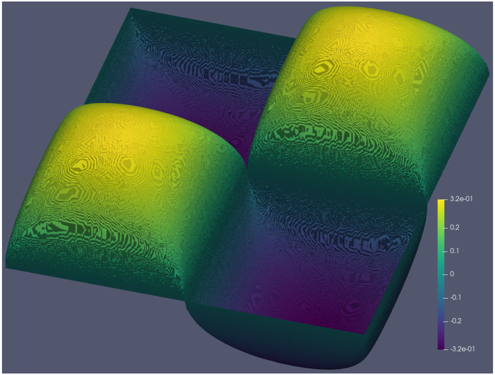
While $\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N} > \delta$:

- Initialize the solution $u_{\mathcal{Q}_s^N,k} = 0$
- Initialize the local Bank–Weiser solutions $\{e^{\mathrm{bw}}_{\mathcal{Q}_s^N,T} = 0\}_T$
- **For** each parametric problem $l \in [\![1, N]\!]$:
  - Solve parametric problem on $\mathcal{T}_n$ to obtain $u_{l,k}$
  - Add $u_{\mathcal{Q}_s^N,k} + C_s(N) a_l u_{l,k}$
  - **For** each cell T of $\mathcal{T}_n$:
    - Solve BW local parametric problem on T to obtain $e^{\mathrm{bw}}_{l,T}$
    - Add $e^{\mathrm{bw}}_{\mathcal{Q}_s^N,T} + C_s(N) a_l e^{\mathrm{bw}}_{l,T}$
- Compute $\{\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N,T} = \|e^{\mathrm{bw}}_{\mathcal{Q}_s^N,T}\|_{L^2(T)}\}_T$
- Take the square root of the sum of $\{{\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N,T}}^2\}_T$ to obtain $\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N}$
- **If** $\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N} > \delta$:
  - Mark the mesh using $\{\eta^{\mathrm{bw}}_{\mathcal{Q}_s^N,T}\}_T$
  - Refine the mesh and replace $\mathcal{T}_n$ by $\mathcal{T}_{n+1}$

The spectral fractional Laplacian

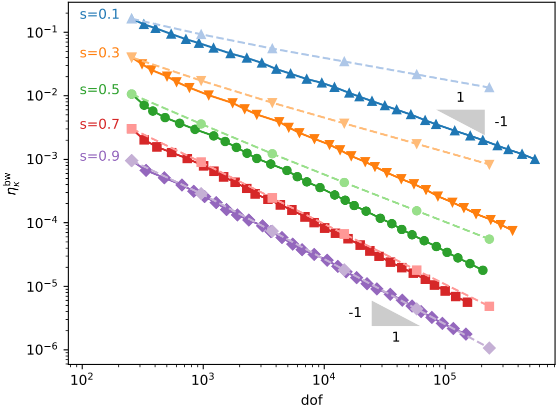
Numerical results

$(-\Delta)^s u = f,$ in $[0,1]^2,\ u = 0,$ on $\Gamma,$
with $f(x,y) = 1$ in $[0,0.5]^2 \cup [0.5,1]^2,\ -1$ otherwise.
We assume the rational approximation is negligible, i.e. $u = u_{\mathcal{Q}_s^N}$.



The spectral fractional Laplacian

Numerical results

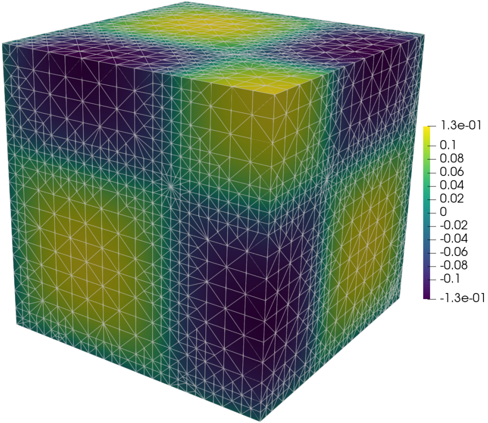


Frac. power	0.1	0.3	0.5	0.7	0.9
Theory [Bonito and Pasciak, 2015]	-0.35	-0.55	-0.75	-0.95	-1.00
Est. (unif.)	-0.35	-0.55	-0.76	-0.95	-1.00
Est. (adapt.)	-0.65	-0.84	-0.93	-0.97	-1.01

The spectral fractional Laplacian

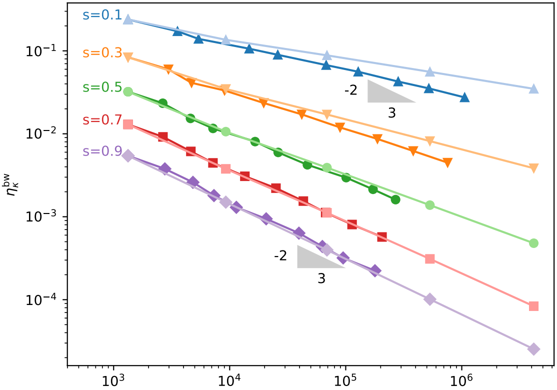
Numerical results

$(-\Delta)^s u = f$, in $[0,1]^3$, $u = 0$, on Γ.
We assume the rational approximation is negligible, i.e. $u = u_{\mathcal{Q}_s^N}$.



The spectral fractional Laplacian

Numerical results



Frac. power	0.1	0.3	0.5	0.7	0.9
Theory [Bonito and Pasciak, 2015]	-0.23	-0.37	-0.50	-0.63	-0.67
Est. (unif.)	-0.24	-0.38	-0.52	-0.62	-0.67
Est. (adapt.)	-0.33	-0.46	-0.55	-0.65	-0.68

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Thank you for your attention!

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