Discretization of the fractional Laplacian using finite element methods and a posteriori error estimation

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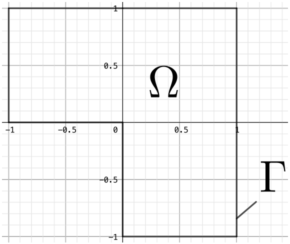


Let $\gamma \in \mathbb{R}^{+,*}$ and $f \in L^2(\Omega)$. We are looking for u (with sufficient regularity) such that

$$\begin{aligned} u - \gamma \Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma, \end{aligned}$$

where $\Delta u(x_1, x_2) := (\partial^2_{xx} + \partial^2_{yy})u$.

A finite element method

Example of elliptic PDE

Instead of looking at the strong formulation: find u satisfying

$$\begin{aligned} u - \gamma \Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma, \end{aligned}$$

A finite element method

Example of elliptic PDE

Instead of looking at the strong formulation: find u satisfying

$$\begin{aligned} u - \gamma \Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma, \end{aligned}$$

we consider the weak formulation: find a function u in $H_0^1(\Omega)$ such that

$$\int_\Omega uv + \gamma \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \quad \forall v \in H_0^1(\Omega),$$

where $H_0^1(\Omega)$ is the Sobolev space of functions v in $L^2(\Omega)$ vanishing on Γ and with $\partial_x v$ and $\partial_y v$ in $L^2(\Omega)$.

A finite element method

Discretization

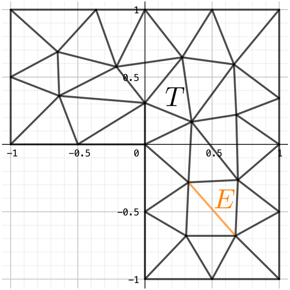
Goal: Compute a numerical approximation to u, solution to

$$\int_\Omega uv + \gamma \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \quad \forall v \in H_0^1(\Omega).$$

A finite element method

Discretization

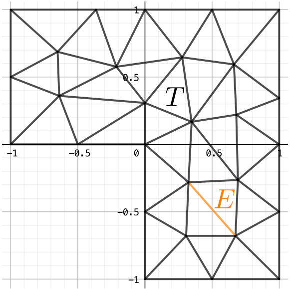
Let $\mathcal{T} = \{T\}$ be a mesh on Ω, of edges $\mathcal{E} = \{E\}$.



A finite element method

Discretization

Let $\mathcal{T} = \{T\}$ be a mesh on Ω, of edges $\mathcal{E} = \{E\}$.
We consider $V^1 \subset H_0^1(\Omega)$ the space of continuous piecewise linear polynomial functions over $\mathcal{T}$, vanishing on Γ.



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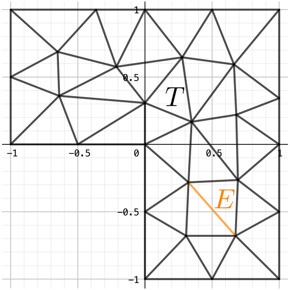

Let $u_1 \in V^1$ be the solution to

$$\int_\Omega u_1 v_1 + \gamma \int_\Omega \nabla u_1 \cdot \nabla v_1 = \int_\Omega f v_1 \quad \forall v_1 \in V^1.$$

A finite element method

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Original problem:

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We take $u_1 \approx u$.

A finite element method

Discretization

Let's try it out!

A finite element method

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We take $\gamma = 1$, $f = 1$ and solve

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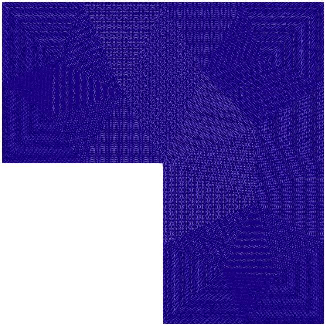

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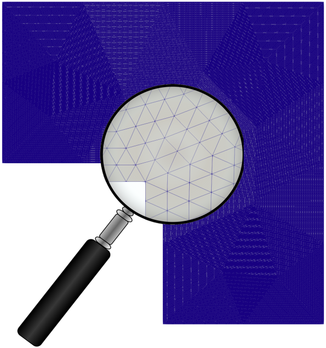

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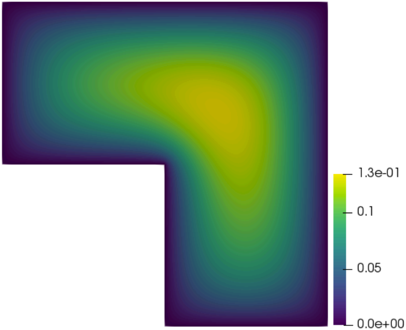
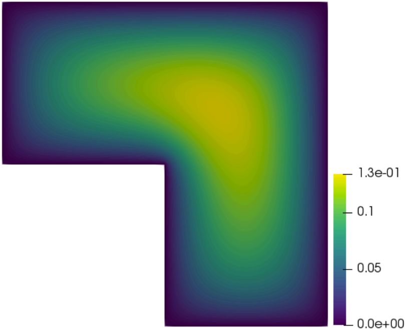
$$\int_\Omega u_1 v_1 + \int_\Omega \nabla u_1 \cdot \nabla v_1 = \int_\Omega f v_1 \quad \forall v_1 \in V^1.$$

(Linear system dimension: 66049.)





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Error estimation

A priori error estimation

What can we say about the discretization error ?

Error estimation

A priori error estimation

What can we say about the discretization error ?
We quantify the error $e := u - u_1$ using the energy norm

$$\|e\|_\gamma := \left(\int_\Omega e^2 + \gamma \int_\Omega \nabla e \cdot \nabla e \right)^{1/2}.$$

Error estimation

A priori error estimation

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A priori error estimation

Let Ω be an open subset of $\mathbb{R}^2$ of polygonal boundary and let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of conformal meshes of Ω. Then,

$$\lim_{h \to 0} \|e\|_\gamma = 0.$$

Moreover, if $u \in H^2(\Omega)$ there exists c_γ such that

$$\|e\|_\gamma \leqslant c_\gamma h |u|_{H^2}.$$

$H^2(\Omega) := \left\{v \in L^2(\Omega),\ \partial^\alpha u \in L^2(\Omega),\ \alpha \in \mathbb{N}^2,\ |\alpha| \leqslant 2\right\}$ is an Hilbert space on which we define the semi-norm, $|u|^2_{H^2} := \|\partial^2_{xx} u\|^2_{L^2} + \|\partial^2_{yy} u\|^2_{L^2} + \|\partial^2_{xy} u\|^2_{L^2}$.

Error estimation

A priori error estimation

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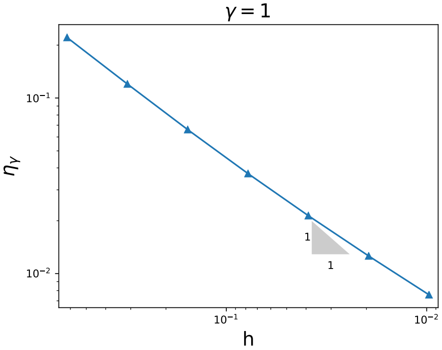
A priori error estimation

$$\|e\|_\gamma \leqslant c_\gamma h^1 |u|_{H^2}.$$

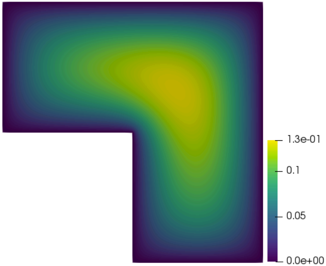


Error estimation

What happened ?

A priori error estimation

$$\|e\|_\gamma \leqslant c_\gamma h|u|_{H^2}.$$



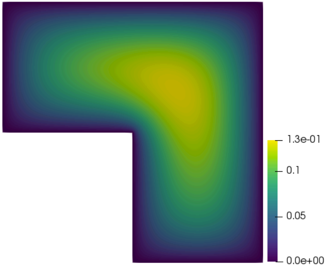
Error estimation

What happened ?

A priori error estimation

$$\|e\|_\gamma \leqslant c_\gamma h |u|_{H^2}.$$

The solution u does not belong to $H^2(\Omega)$!
∇u admits a singularity in the reentrant corner of Ω [Grisvard, 1986].



Error estimation

A posteriori error estimation

- How to deal with solutions having local features ?

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Error estimation

A posteriori error estimation

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A priori error estimation	A posteriori error estimation
$\|e\|_\gamma \leqslant \widetilde{C}(u)$	$\|e\|_\gamma \approx \eta$
$\widetilde{C}(u)$ is unknown.	η is known.

Error estimation

A posteriori error estimation

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- computable from problem data (f, boundary conditions data...) and u_1 only,

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- efficient i.e. there exists a constant c only depending on the mesh regularity such that

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$$\eta = \sum_{T\in\mathcal{T}} \eta_T,$$

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- cheap to compute, ideally much less expensive than computing u_1.

Error estimation

A posteriori error estimation

Let $e := u - u_1$, we can show that e is solution to the local problem:

$$\int_T e v_T + \gamma \int_T \nabla e \cdot \nabla v_T = \int_T r_{\gamma,T} v_T + \sum_{E \in \partial T} \int_E J_{\gamma,E} v_T \quad \forall v_T \in H_0^1(T),$$

with $r_{\gamma,T} := (f - u_1 + \gamma \Delta u_1)_{|T}$ and $J_{\gamma,E} := \gamma \left[\!\!\left[\frac{\partial u_1}{\partial n} \right]\!\!\right]_E$.

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We consider $V^{\mathrm{bw}}(T) \subset V^2(T)$ a particular local finite element space.
For a cell T in $\mathcal{T}$, we define $e_T^{\mathrm{bw}} \in V^{\mathrm{bw}}(T)$ the solution to

$$\int_T e_T^{\mathrm{bw}} v_T + \gamma \int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T = \int_T r_{\gamma,T} v_T + \sum_{E \in \partial T} \int_E J_{\gamma,E} v_T \quad \forall v_T \in V^{\mathrm{bw}}(T),$$

Error estimation

A posteriori error estimation

The local Bank-Weiser a posteriori error estimator [Bank, Weiser, 1985] is defined by:

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Error estimation

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The local Bank-Weiser a posteriori error estimator [Bank, Weiser, 1985] is defined by:

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and the global estimator by:

$$\eta_{\mathrm{bw}}^2 := \sum_{T \in \mathcal{T}} \eta_T^2.$$

Error estimation

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Is the Bank-Weiser estimator a good estimator ?

- computable from data and u_1 only: ✓

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Is the Bank-Weiser estimator a good estimator ?

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- efficient ($c\eta_{\mathrm{bw}} \leqslant \|e\|_\gamma$): ✓ [Bank, Weiser, 1985],
- local: ✓
- cheap: ✓ [Bordas, B., Chouly, Hale, Lozinski, 2020].

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Numerical results

Let's try it out!

Adaptive refinement algorithm:

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7. go back to 2. replacing l by $l + 1$.

Numerical results

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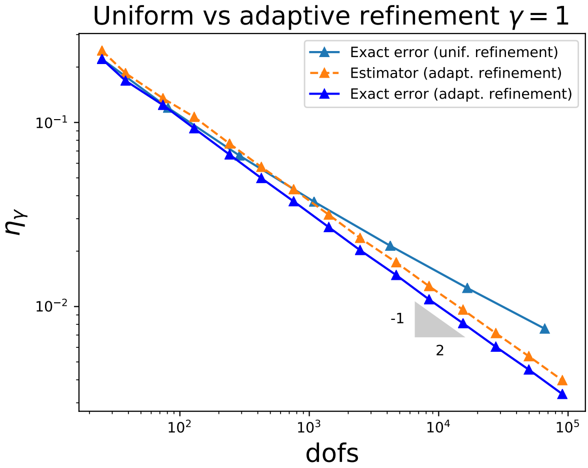

Numerical results

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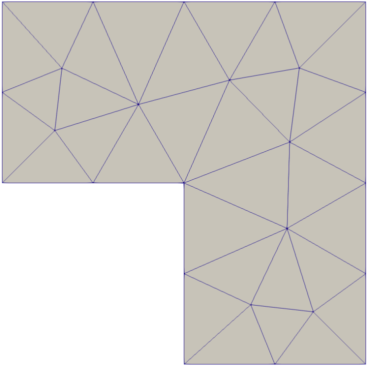
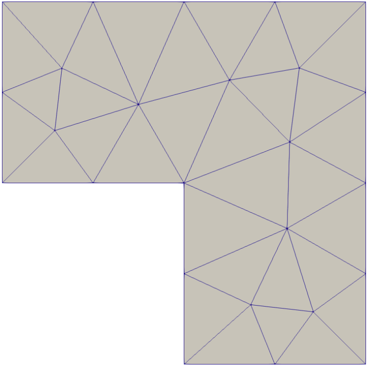
Uniform refinement:
Initial mesh



Adaptive refinement:
Initial mesh



Exact error ≈ 0.2210
Linear system dim. $= 25$

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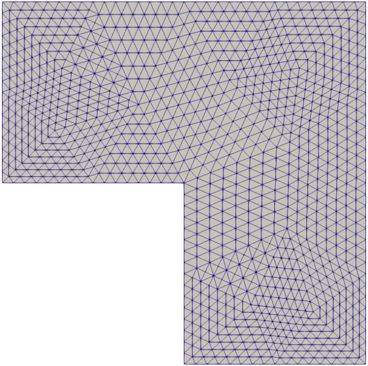
Numerical results

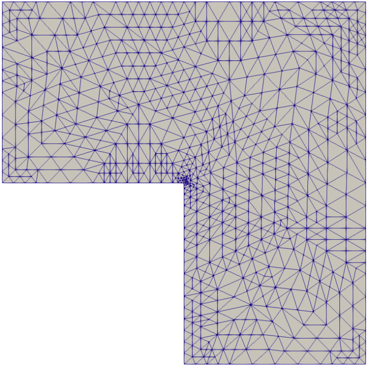
Let's try it out!

Uniform refinement: 3rd mesh



Exact error ≈ 0.0371
Linear system dim. $= 1089$

Adaptive refinement: 6th mesh

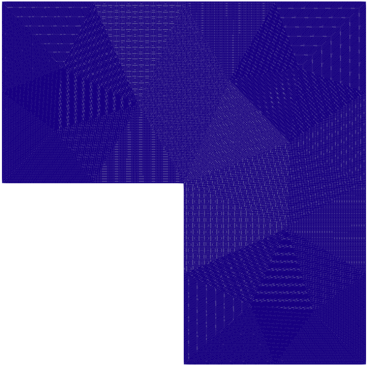


Exact error ≈ 0.0372
Linear system dim. $= 757$

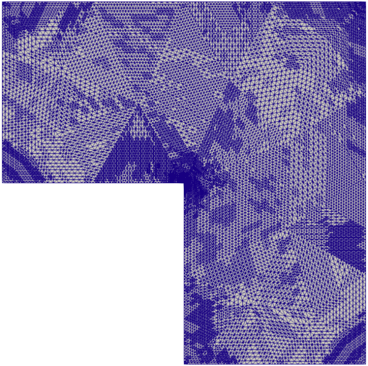
Numerical results

Let's try it out!

Uniform refinement:
6th mesh



Adaptive refinement:
11th mesh



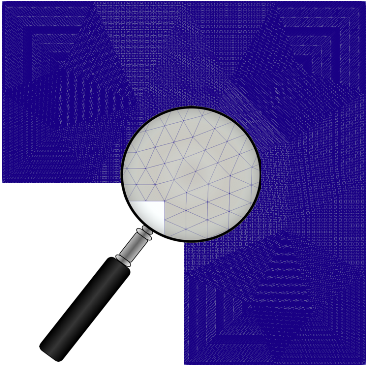
Exact error ≈ 0.0076
Linear system dim. $= 66049$

Exact error ≈ 0.0081
Linear system dim. $= 15429$

Numerical results

Let's try it out!

Uniform refinement:
6th mesh



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Application to a fractional Laplacian problem

The spectral fractional Laplacian

Let $\alpha \in (0, 2)$ and $f \in L^2(\Omega)$, we are looking for $u \in L^2(\Omega)$ (with sufficient regularity) such that

$$\begin{aligned} (-\Delta)^{\alpha/2} u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned}$$

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How is the function u defined ?

Let $\mathcal{L}$ be the Laplace-Dirichlet operator on Ω such that $\mathcal{L} w = f$ if w is the solution of

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We consider the weak formulation: w in $H_0^1(\Omega)$ is solution to

$$\int_\Omega \nabla w \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega).$$

Application to a fractional Laplacian problem

The spectral fractional Laplacian

The eigenfunctions $\{\psi_j\}_{j=1}^{\infty}$ of $\mathcal{L}$ form a basis of $L^2(\Omega)$.

$$f = \sum_{j=1}^{\infty} f_j \psi_j,$$

with $f_j := \int_\Omega f \psi_j$ for $j = 1, \cdots, \infty$.

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with $f_j := \int_\Omega f\psi_j$ for $j = 1, \cdots, \infty$.
If $\{\lambda_j\}_{j=1}^{\infty}$ are the corresponding eigenvalues, we define the solution u as follow:

$$u = (-\Delta)^{-\alpha/2} f = \mathcal{L}^{-\alpha/2} f := \sum_{j=1}^{\infty} \lambda_j^{-\alpha/2} f_j \psi_j.$$

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$$\frac{\pi}{\sin(\pi\theta)} = \int_0^{+\infty} t^{\theta-1}(1+t)^{-1}\,\mathrm{d}t \quad \forall\theta \in (0,1).$$

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Some tweaks lead to

$$s^{\theta-1} = c_\theta \int_0^{+\infty} t^{\theta-1}(s+t)^{-1}\ \mathrm{d}t \quad \forall s > 0 \text{ and } \forall \theta \in (0,1),$$

with $c_\theta = \frac{\sin(\pi\theta)}{\pi}$.

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with $c_\theta = \frac{\sin(\pi\theta)}{\pi}$.

Then, for $\theta - 1 = -\alpha/2 \in (-1,0)$ and $s = \lambda_j$, $j \in [\![1,+\infty[\![$,

$$\lambda_j^{-\alpha/2} = c_\alpha \int_0^{+\infty} t^{-\alpha/2}(\lambda_j+t)^{-1}\,\mathrm{d}t.$$

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$$\mathcal{L}^{-\alpha/2} f = c_\alpha \int_0^{+\infty} t^{-\alpha/2} (\mathcal{L} + t\,\mathrm{Id})^{-1} f \;\mathrm{d}t,$$

and with a nice change of variable,

$$\mathcal{L}^{-\alpha/2} f = c_\alpha \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha y} \left(\mathrm{Id} + \mathrm{e}^{2y}\,\mathcal{L}\right)^{-1} f \;\mathrm{d}y \quad \text{for } \alpha \in (0,2).$$

with $c_\alpha := \frac{2\sin(\pi\alpha/2)}{\pi}$.

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$$u = \mathcal{L}^{-\alpha/2} f = c_\alpha \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha y} \left(\mathrm{Id} + \mathrm{e}^{2y}\,\mathcal{L}\right)^{-1} f \,\mathrm{d}y \quad \text{for } \alpha \in (0,2).$$

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Let us denote $u_y := (\mathrm{Id} + \mathrm{e}^{2y}\,\mathcal{L})^{-1} f$. This function is solution to the following problem (in weak formulation)

$$\int_\Omega u_y v + \mathrm{e}^{2y} \int_\Omega \nabla u_y \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega).$$

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We want to discretize the above integral into a finite sum involving computable terms.

How to compute the solution numerically ?

Euler's reflection formula

$$c_\alpha \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha y}\, u_y \,\mathrm{d}y = u \approx u_1^N := c_\alpha \sum_{l=-N}^{N} \omega_l\, \mathrm{e}^{\alpha y_l}\, u_{y_l,1}.$$

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1. Using FEM, we discretize the function u_y into $u_{y,1}$ solution to

$$\int_\Omega u_{y,1} v_{y,1} + \mathrm{e}^{2y} \int_\Omega \nabla u_{y,1} \cdot \nabla v_{y,1} = \int_\Omega f v_{y,1} \quad \forall v_{y,1} \in V^1.$$

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2. We discretize the integral using a simple rectangle quadrature rule of weights $\omega_l = \frac{1}{\sqrt{N}}$ and points $y_l = \frac{l}{\sqrt{N}}$ for any $l \in [\![-N, \cdots, N]\!]$, where N is a user chosen parameter.

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A posteriori error estimation

Euler's reflection formula

We would like to quantify the approximation error:

$$\|e\|_{L^2} := \|u - u_1^N\|_{L^2} := \left\| c_\alpha \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha y}\, u_y \mathrm{d}y - \frac{c_\alpha}{\sqrt{N}} \sum_{l=-N}^{N} \mathrm{e}^{\alpha y_l}\, u_{y_l,1} \right\|_{L^2}.$$

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We have,

$$\begin{aligned} \|u - u_1^N\|_{L^2} &= \|u - u_1 + u_1 - u_1^N\|_{L^2} \\ &\leqslant \underbrace{\|u - u_1\|_{L^2}}_{\text{FE error}} + \underbrace{\|u_1 - u_1^N\|_{L^2}}_{\text{quadrature error}}. \end{aligned}$$

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Finite element error

We want to quantify the finite element error $\|u - u_1\|_{L^2}$.

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We want to quantify the finite element error $\|u - u_1\|_{L^2}$.

$$u - u_1 = c_\alpha \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha y}\, u_y \mathrm{d}y - c_\alpha \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha y}\, u_{y,1} \mathrm{d}y$$
$$= c_\alpha \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha y} (u_y - u_{y,1}) \mathrm{d}y.$$

Idea: We already know an estimator for the error $u_y - u_{y,1}$.
Can we use it to estimate $u - u_1$?

A posteriori error estimation

Finite element error

For a fixed y in $\mathbb{R}$ and for a cell T of the mesh, we compute $e^{\mathrm{bw}}_{y,T} \in V^{\mathrm{bw}}(T)$ the solution to

$$\int_T e^{\mathrm{bw}}_{y,T} v_T + \mathrm{e}^{2y} \int_T \nabla e^{\mathrm{bw}}_{y,T} \cdot \nabla v_T = \int_T r_{y,T} v_T + \sum_{E \in \partial T} \int_E J_{y,E} v_T \quad \forall v_T \in V^{\mathrm{bw}}(T).$$

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We compute

$$e^{\mathrm{bw}}_T := c_\alpha \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha y}\, e^{\mathrm{bw}}_{y,T} \mathrm{d}y,$$

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Finite element error

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We compute

$$e_T^{\mathrm{bw}} := c_\alpha \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha y} \, e_{y,T}^{\mathrm{bw}} \mathrm{d}y,$$

$$\eta_{\mathrm{bw},T} := \| e_T^{\mathrm{bw}} \|_{L^2} \quad \text{and} \quad \eta_{\mathrm{bw}}^2 := \sum_{T \in \mathcal{T}} \eta_{\mathrm{bw},T}^2.$$

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$$\|u - u_1\|_{L^2} \approx \eta_{\mathrm{bw}}?$$

Numerical results

Let's try it out!

Adaptive refinement algorithm:

1. fix a tolerance ε, pick an initial (coarse) mesh $\mathcal{T}_j$ $(j = 0)$ and pick a very fine quadrature rule $\{y_l\}_{l=-N}^{N}$ (take N large),

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3. sum the functions $u^j_{y_l,1}$ into the quadrature rule to get u^j_1,

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Numerical results

Let's try it out!

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6. check if $\eta^j_{\mathrm{bw}} \leqslant \varepsilon$,

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   - ✗ continue,

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   - ✗ continue,
7. mark the cells we need to refine (e.g. each cell T for which $\eta^j_{\mathrm{bw},T} \geqslant 0.9 \max_{T\in\mathcal{T}_j} \eta^j_{\mathrm{bw},T}$),

Numerical results

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8. refine the mesh $\mathcal{T}_j$ into $\mathcal{T}_{j+1}$,
9. go back to 2. replacing j by $j + 1$.

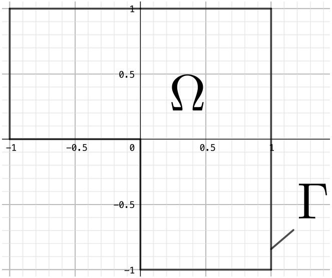
Numerical results

Let's try it out!

Taking $f = 1$, we solve

$$(-\Delta)^{\alpha/2} u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \Gamma.$$

using finite elements and Euler's reflection formula and we estimate the error using the Bank-Weiser estimator.



Numerical results

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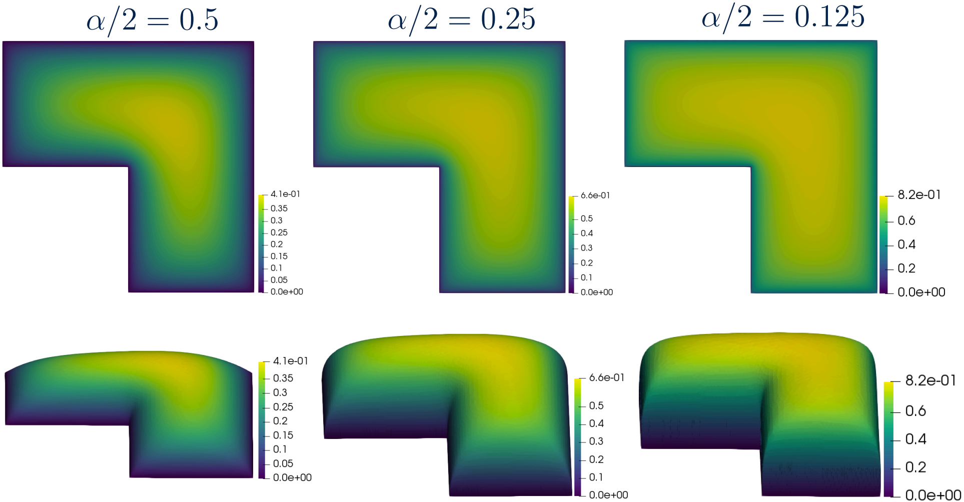

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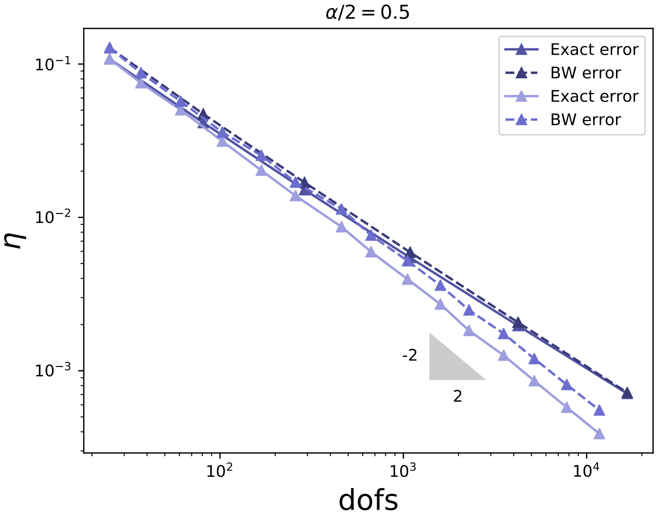

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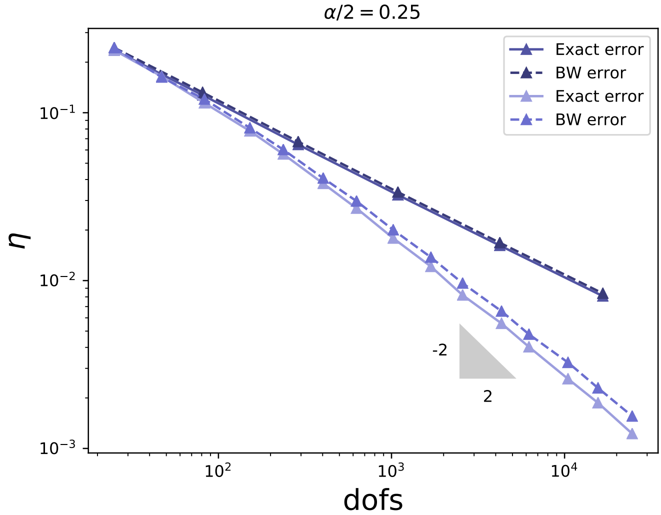

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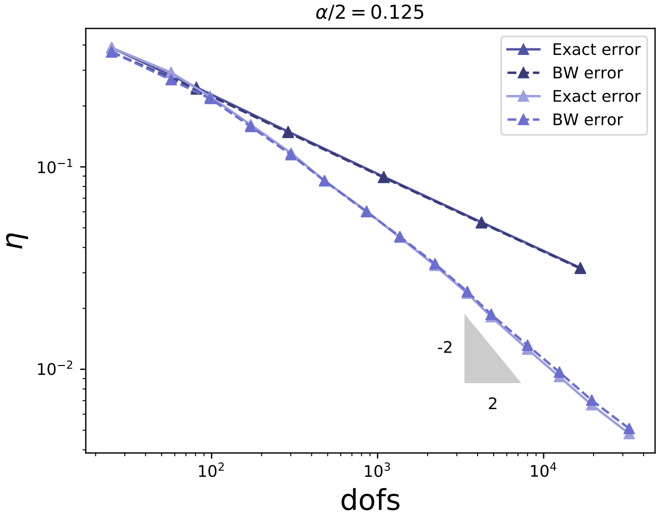
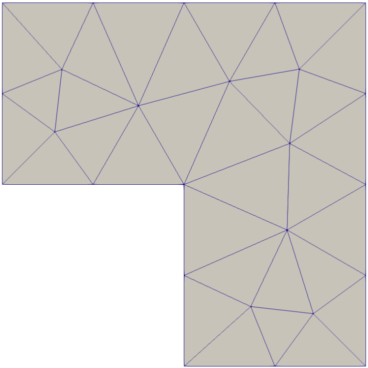


Numerical results

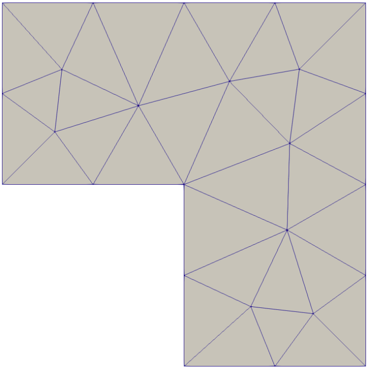
Let's try it out!

$\alpha/2 = 0.5$

Uniform refinement:
Initial mesh



Adaptive refinement:
Initial mesh



Exact error ≈ 0.1079
Linear system dim. $= 25$

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Numerical results

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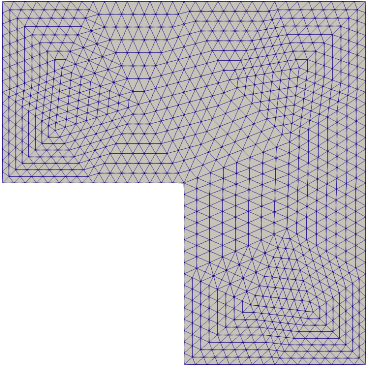
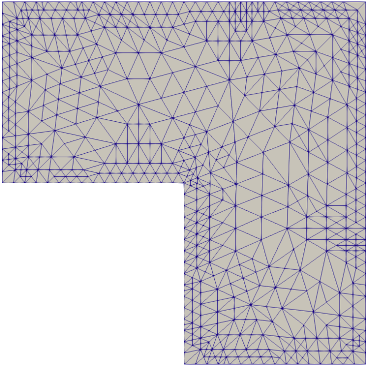
$$\alpha/2 = 0.5$$

Uniform refinement:
3rd mesh



Adaptive refinement:
7th mesh



Exact error ≈ 0.0055
Linear system dim. $= 1089$

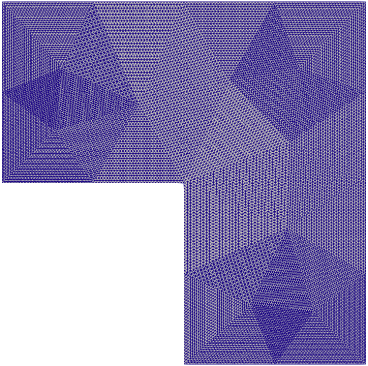
Exact error ≈ 0.0060
Linear system dim. $= 667$

Numerical results

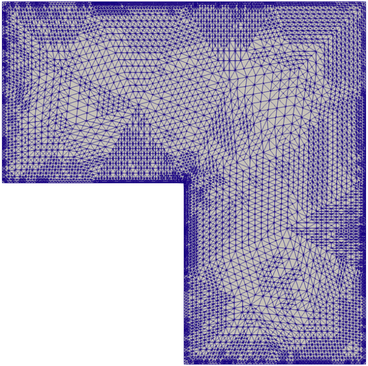
Let's try it out!

$\alpha/2 = 0.5$

Uniform refinement:
5th mesh



Adaptive refinement:
13th mesh



Exact error ≈ 0.0007
Linear system dim. $= 16641$

Exact error ≈ 0.0006
Linear system dim. $= 7791$

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- Become famous and get a permanent job.

Thank you for your attention!