A POSTERIORI ERROR ESTIMATION FOR THE FRACTIONAL LAPLACIAN

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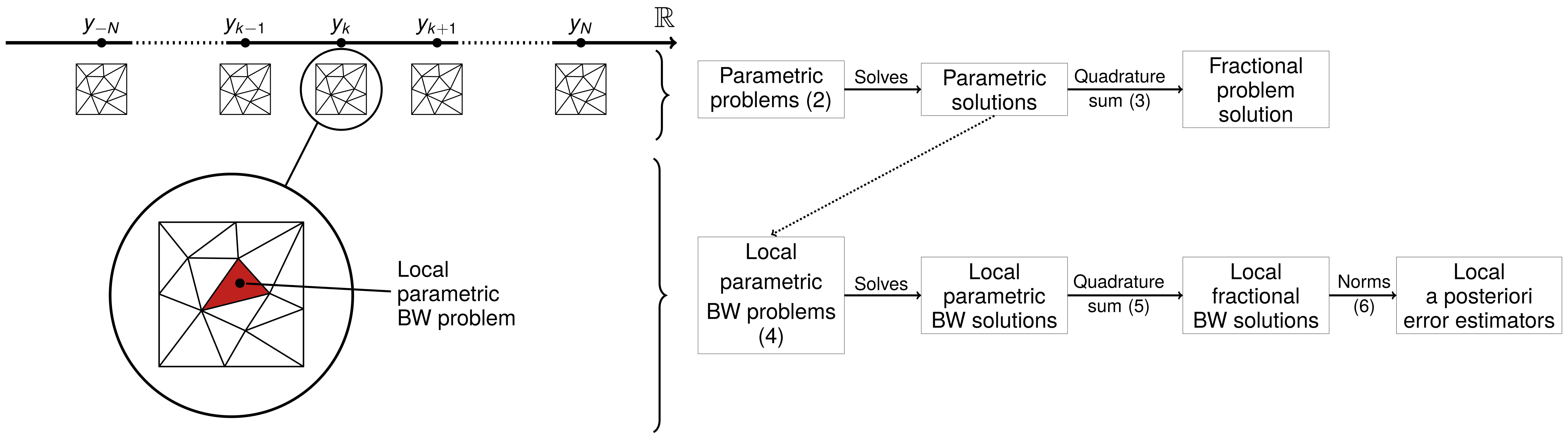

The method in [Bonito and Pasciak, 2013] solves fractional elliptic operator equations by re-writing the operator as an integral over solutions of standard parametric elliptic problems. **Can we use a similar idea to derive an a posteriori error estimator for the spatial discretization of the method in [Bonito and Pasciak, 2013] ?**

CONTRIBUTIONS

- We derive a sharp **a posteriori error estimator** for the finite element discretization of fractional Laplacian PDEs.
- We perform **adaptive mesh refinement**.
- We use the **FEniCS project** and our a posteriori error estimation **package FEniCS-EE** [Hale and Bulle, 2020].

FRACTIONAL PROBLEM

For any $\alpha \in (0,2)$, $d = 1, 2$ or 3 and $f \in L^2(\Omega)$, we consider the fractional Laplacian equation on a polygonal domain Ω in $\mathbb{R}^d$

$$(-\Delta)^{\alpha/2} u = f \text{ in } \Omega, \quad u = 0 \text{ in } \partial\Omega. \tag{1}$$

The solution u can be represented by

$$u = C_\alpha \int_{-\infty}^{+\infty} e^{\alpha y} u_y \, dy,$$

where C_α is a constant depending on α and u_y is the solution of the parametric problem

$$\int_\Omega u_y v + e^{2y} \int_\Omega \nabla u_y \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega).$$

FINITE ELEMENT DISCRETIZATION

The discretization of (1) relies on two things [Bonito and Pasciak, 2013]:

- **A finite element method**: to discretize the parametric problems. Let $\mathcal{T}$ be a triangulation on Ω and $V^1 \subset H_0^1(\Omega)$ be the linear Lagrange finite elements space on $\mathcal{T}$.

$$u = C_\alpha \int_{-\infty}^{+\infty} e^{\alpha y} u_y \, dy \approx C_\alpha \int_{-\infty}^{+\infty} e^{\alpha y} u_{1,y} \, dy =: u_1,$$

where $u_{1,y}$ is the solution of the parametric finite element problem

$$\int_\Omega u_{1,y} v_1 + e^{2y} \int_\Omega \nabla u_{1,y} \cdot \nabla v_1 = \int_\Omega f v_1 \quad \forall v_1 \in V^1. \tag{2}$$

Note: the same mesh is used for every parametric problems.

- **A quadrature method**: to discretize the integral over y. Let $\{\omega_k, y_k\}_{k=-N}^N$ be a quadrature rule on $\mathbb{R}$.

$$u_1 := C_\alpha \int_{-\infty}^{+\infty} e^{\alpha y} u_{1,y} \, dy \approx C_\alpha \sum_{k=-N}^{N} \omega_k \, e^{\alpha y_k} \, u_{1,y_k} =: u_1^N, \tag{3}$$

A POSTERIORI ERROR ESTIMATION

We are interested in the **spatial discretization error only** so we consider the quadrature error to be negligible.

We would like to estimate the error $\|u - u_1\|_{L^2(T)}$ on each cell T of the mesh.

Given $V^1(T)$ and $V^2(T)$ respectively the local linear and local quadratic Lagrange finite elements spaces and $\mathcal{L}_T : V^2(T) \longrightarrow V^1(T)$ the Lagrange interpolation operator, we consider $V^{\mathrm{bw}}(T) = \{v \in V^2(T), \ \mathcal{L}_T(v) = 0\}$ [Bank and Weiser, 1985].

- For each paramteric problem (2) we derive the local Bank–Weiser (BW) problem given by

$$\int_T e_{T,y}^{\mathrm{bw}} v^{\mathrm{bw}} + e^{2y} \int_T \nabla e_{T,y}^{\mathrm{bw}} \cdot \nabla v^{\mathrm{bw}} = \int_T r_y v^{\mathrm{bw}} + \frac{1}{2} \sum_{E \in \partial T} \int_E J_y v^{\mathrm{bw}} \quad \forall v^{\mathrm{bw}} \in V^{\mathrm{bw}}(T), \tag{4}$$

where $r_T := f - u_{1,y} + e^{2y} \Delta u_{1,y}$ and $J_y := e^{2y} \left[\!\left[\frac{\partial u_{1,y}}{\partial n} \right]\!\right]$.

- We sum the solutions $\{e_{T,y_k}^{\mathrm{bw}}\}_{k=-N}^N$ using the same quadrature rule as (3)

$$e_T^{\mathrm{bw}} := C_\alpha \sum_{k=-N}^{N} \omega_k e^{\alpha y_k} e_{T,y_k}^{\mathrm{bw}}. \tag{5}$$

- We take the norms of the local functions $\{e_T^{\mathrm{bw}}\}_{T \in \mathcal{T}}$ to get the **local BW estimators**

$$\|e_T^{\mathrm{bw}}\|_{L^2(T)} =: \eta_T^{\mathrm{bw}}. \tag{6}$$

NUMERICAL RESULTS

We solve (1) on $\Omega = (0,1)^2$ for $\alpha = 0.5$ and $f = 1$ on $(0, 0.5)^2 \cup (0.5, 1)^2$ and $f = -1$ on $(0, 0.5) \times (0.5, 1) \cup (0.5, 1) \times (0, 0.5)$. We compare uniform and adaptive mesh refinement. The convergence curves for the L^2 error (computed from the comparison with a higher order FE discretization) and the estimator are shown in Fig.5.

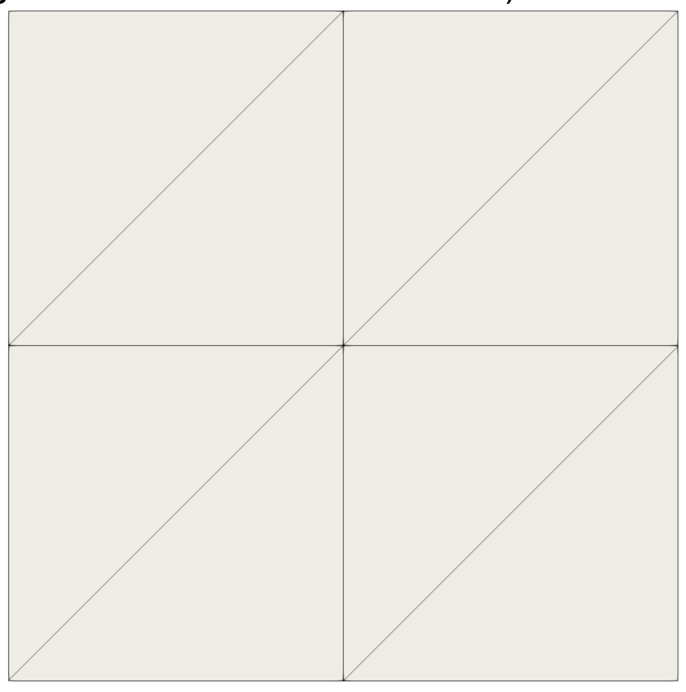

Fig.1 Initial mesh.

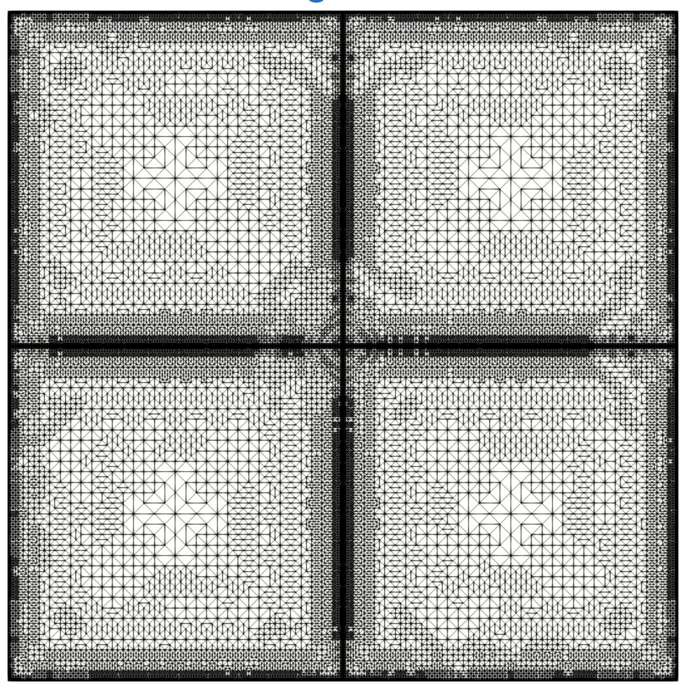

Fig.2 Final mesh after 17 adapt. ref. steps.

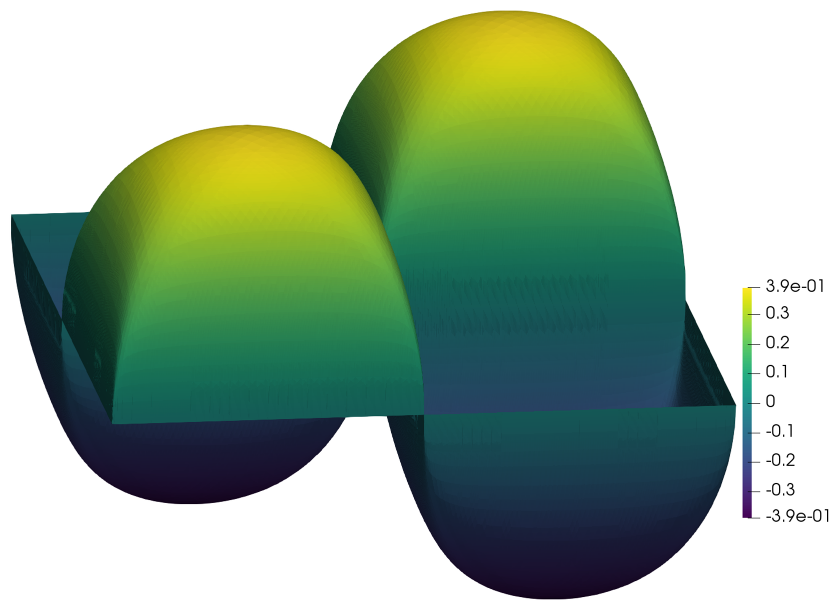

Fig.3 Solution u after 17 adapt. ref. steps.

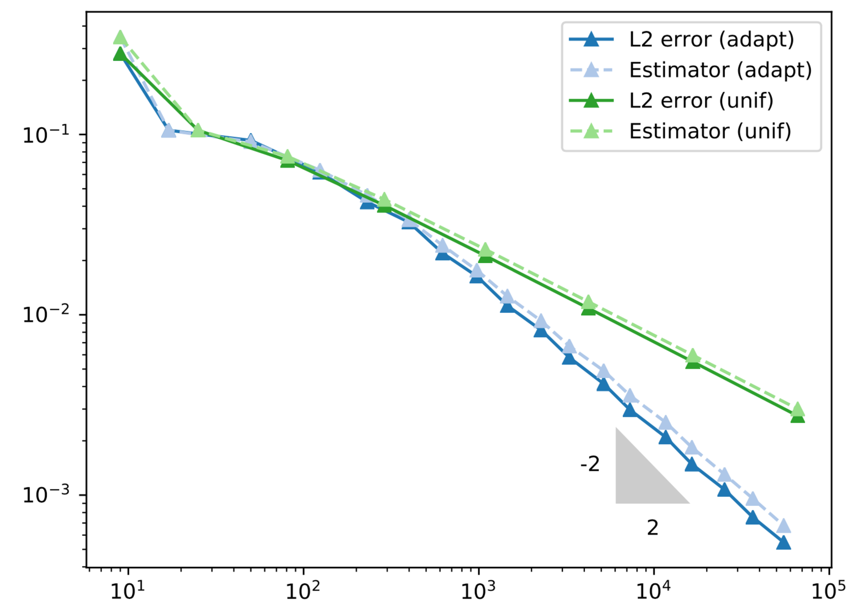

Fig.4 Convergence curves comparison.

PRELIMINARY FINDINGS

- Numerical evidences show that the error **estimation is sharp**.
- The algorithm is **perfectly parallelizable**.
- It works the same way for **1, 2 or 3D** problems and for **higher order finite elements**.

FUTURE WORK

- Use this method on **parabolic** fractional linear equations [Bonito, Lei, and Pasciak, 2016; Bonito, Lei, and Pasciak, 2017].
- Adapt the method to **other boundary conditions** [Antil, Pfefferer, and Rogovs, 2018].
- Try to prove the **reliability and efficiency** of the estimator.
- Apply it to the **study of non-local physical phenomena** as non-local diffusion.

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