Hierarchical a posteriori error estimation in the FEniCS finite element software and applications to fractional PDEs.

Raphaël Bulle

Stéphane P.A. Bordas, Jack S. Hale,

Franz Chouly, Alexei Lozinski

University of Luxembourg
Université de Bourgogne Franche-Comté

December 9, 2021

Background

2010-2013 **Bachelor in Mathematics**

at Université de Bourgogne Franche-Comté (FR).

Background

2014 **CAPES** (competitive exam)

of Mathematics.

2010-2013 **Bachelor in Mathematics**

at Université de Bourgogne Franche-Comté (FR).

Background

2016 **Agrégation** (competitive exam)

of Mathematics.

2014 **CAPES** (competitive exam)

of Mathematics.

2010-2013 **Bachelor in Mathematics**

at Université de Bourgogne Franche-Comté (FR).

Background

2015-2017 **Master in Advanced Mathematics**

at Université de Bourgogne Franche-Comté.

2016 **Agrégation** (competitive exam)

of Mathematics.

2014 **CAPES** (competitive exam)

of Mathematics.

2010-2013 **Bachelor in Mathematics**

at Université de Bourgogne Franche-Comté (FR).

Background

2017-2022 **PhD student in Computational Engineering and Applied Mathematics**
at University of Luxembourg and Université de Bourgogne Franche-Comté

Supervision: S. P. A. Bordas, F. Chouly, J. S. Hale and A. Lozinski.

2015-2017 **Master in Advanced Mathematics**
at Université de Bourgogne Franche-Comté.

2016 **Agrégation** (competitive exam)
of Mathematics.

2014 **CAPES** (competitive exam)
of Mathematics.

2010-2013 **Bachelor in Mathematics**
at Université de Bourgogne Franche-Comté (FR).

Table of contents

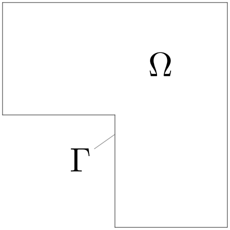
- Background
- The Bank–Weiser estimator
  - Toy problem setting
  - Definition
  - Properties
- Implementation
  - Method details
  - Numerical results
- The spectral fractional Laplacian
  - Problem setting
  - Discretization
- A posteriori error estimation
  - Rational approximation error
  - Finite element error
- Numerical results

The Bank–Weiser estimator

Toy problem setting

Let $f \in L^2(\Omega)$, we look for u with sufficient regularity s.t.

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma.$$



The Bank–Weiser estimator

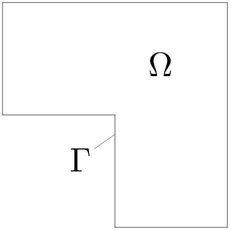
Toy problem setting

Let $f \in L^2(\Omega)$, we look for u with sufficient regularity s.t.

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma.$$

In weak formulation, find u in $H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v, \quad \forall v \in H^1_0(\Omega).$$

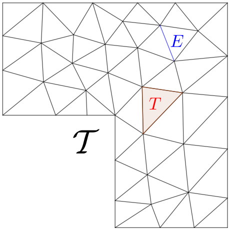


The Bank–Weiser estimator

Toy problem setting

Let $f \in L^2(\Omega)$, we look for u with sufficient regularity s.t.

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma.$$



In weak formulation, find u in $H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v, \quad \forall v \in H_0^1(\Omega).$$

Lagrange finite element discretization of order k, find u_k in V^k such that

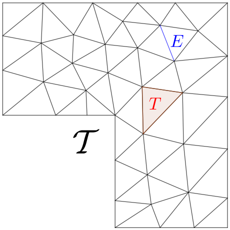
$$\int_\Omega \nabla u_k \cdot \nabla v_k = \int_\Omega f v_k, \quad \forall v_k \in V^k.$$

The Bank–Weiser estimator

Toy problem setting

Let $f \in L^2(\Omega)$, we look for u with sufficient regularity s.t.

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma.$$



In weak formulation, find u in $H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v, \quad \forall v \in H_0^1(\Omega).$$

Lagrange finite element discretization of order k, find u_k in V^k such that

$$\int_\Omega \nabla u_k \cdot \nabla v_k = \int_\Omega f v_k, \quad \forall v_k \in V^k.$$

Goal: estimate $\eta_{\text{err}} = \|\nabla(u_k - u)\|_\Omega$ i.e. find a computable quantity η_{bw} such that $\eta_{\text{bw}} \approx \eta_{\text{err}}$.

The Bank–Weiser estimator

Definition

On a cell T, the Bank–Weiser problem is given by:
find e_T^{bw} in V_T^{bw} such that

$$\int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T^{\mathrm{bw}} = \int_T r_T v_T^{\mathrm{bw}} + \sum_{E \in \partial T} \frac{1}{2} \int_E J_E v_T^{\mathrm{bw}} \qquad \forall v_T^{\mathrm{bw}} \in V_T^{\mathrm{bw}}.$$

The Bank–Weiser estimator

Definition

On a cell T, the Bank–Weiser problem is given by:
find e_T^{bw} in V_T^{bw} such that

$$\int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T^{\mathrm{bw}} = \int_T r_T v_T^{\mathrm{bw}} + \sum_{E \in \partial T} \frac{1}{2} \int_E J_E v_T^{\mathrm{bw}} \qquad \forall v_T^{\mathrm{bw}} \in V_T^{\mathrm{bw}}.$$

The Bank–Weiser estimator is defined as

$$\eta_{\mathrm{bw}}^2 := \sum_{T \in \mathcal{T}} \eta_{\mathrm{bw},T}^2, \quad \eta_{\mathrm{bw},T} := \|\nabla e_T^{\mathrm{bw}}\|_T.$$

The Bank–Weiser estimator

Definition

How is V_T^{bw} defined ?

Let $V_T^- \subsetneq V_T^+$ be two finite element spaces and

$$\mathcal{L}_T : V_T^+ \longrightarrow V_T^-,$$

be the local Lagrange interpolation operator,

$$V_T^{\mathrm{bw}} := \ker(\mathcal{L}_T) = \left\{ v_T^+ \in V_T^+,\ \mathcal{L}_T(v_T^+) = 0 \right\}.$$

The Bank–Weiser estimator

Definition

How is V_T^{bw} defined ?

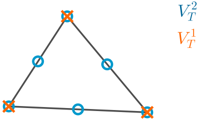
Let $V_T^- \subsetneq V_T^+$ be two finite element spaces and

$$\mathcal{L}_T : V_T^+ \longrightarrow V_T^-,$$

be the local Lagrange interpolation operator,

$$V_T^{\mathrm{bw}} := \ker(\mathcal{L}_T) = \left\{ v_T^+ \in V_T^+,\ \mathcal{L}_T(v_T^+) = 0 \right\}.$$

Examples:



The Bank–Weiser estimator

Definition

How is V_T^{bw} defined ?

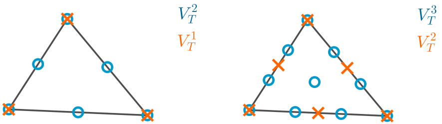
Let $V_T^- \subsetneq V_T^+$ be two finite element spaces and

$$\mathcal{L}_T : V_T^+ \longrightarrow V_T^-,$$

be the local Lagrange interpolation operator,

$$V_T^{\mathrm{bw}} := \ker(\mathcal{L}_T) = \left\{ v_T^+ \in V_T^+,\ \mathcal{L}_T(v_T^+) = 0 \right\}.$$

Examples:



The Bank–Weiser estimator

Definition

How is V_T^{bw} defined ?

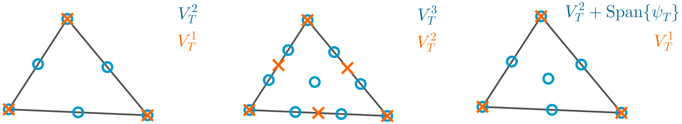
Let $V_T^- \subsetneq V_T^+$ be two finite element spaces and

$$\mathcal{L}_T : V_T^+ \longrightarrow V_T^-,$$

be the local Lagrange interpolation operator,

$$V_T^{\mathrm{bw}} := \ker(\mathcal{L}_T) = \left\{ v_T^+ \in V_T^+,\ \mathcal{L}_T(v_T^+) = 0 \right\}.$$

Examples:



The Bank–Weiser estimator

Properties

- Efficiency ($\eta_{\mathrm{bw}} \leqslant C\eta_{\mathrm{err}} + \mathrm{h.o.t.}$):

The Bank–Weiser estimator

Properties

- Efficiency ($\eta_{\mathrm{bw}} \leqslant C\eta_{\mathrm{err}} + \mathrm{h.o.t.}$):
  - ✓ global [Bank and Weiser, 1985], local [Nochetto, 1993, Verfürth, 1994].

The Bank–Weiser estimator

Properties

- Efficiency ($\eta_{\mathrm{bw}} \leqslant C\eta_{\mathrm{err}} + \mathrm{h.o.t.}$):
  - ✓ global [Bank and Weiser, 1985], local [Nochetto, 1993, Verfürth, 1994].
- Reliability ($\eta_{\mathrm{err}} \leqslant C\eta_{\mathrm{bw}} + \mathrm{h.o.t.}$):

The Bank–Weiser estimator

Properties

- Efficiency ($\eta_{\mathrm{bw}} \leqslant C\eta_{\mathrm{err}} + \mathrm{h.o.t.}$):
  - ✓ global [Bank and Weiser, 1985], local [Nochetto, 1993, Verfürth, 1994].
- Reliability ($\eta_{\mathrm{err}} \leqslant C\eta_{\mathrm{bw}} + \mathrm{h.o.t.}$):
  - ✓ without restriction on k, V_T^+ or V_T^- but under a saturation assumption [Bank and Weiser, 1985],

The Bank–Weiser estimator

Properties

- Efficiency ($\eta_{\mathrm{bw}} \leqslant C\eta_{\mathrm{err}} + \mathrm{h.o.t.}$):
  - ✓ global [Bank and Weiser, 1985], local [Nochetto, 1993, Verfürth, 1994].
- Reliability ($\eta_{\mathrm{err}} \leqslant C\eta_{\mathrm{bw}} + \mathrm{h.o.t.}$):
  - ✓ without restriction on k, V_T^+ or V_T^- but under a saturation assumption [Bank and Weiser, 1985],
  - ✓ without the saturation assumption but under some restrictions either on the space V^{bw} or on the spatial dimension [Nochetto, 1993, Verfürth, 1994],

The Bank–Weiser estimator
Properties

- Efficiency ($\eta_{\mathrm{bw}} \leqslant C\eta_{\mathrm{err}} + \mathrm{h.o.t.}$):
  - ✓ global [Bank and Weiser, 1985], local [Nochetto, 1993, Verfürth, 1994].
- Reliability ($\eta_{\mathrm{err}} \leqslant C\eta_{\mathrm{bw}} + \mathrm{h.o.t.}$):
  - ✓ without restriction on k, V_T^+ or V_T^- but under a saturation assumption [Bank and Weiser, 1985],
  - ✓ without the saturation assumption but under some restrictions either on the space V^{bw} or on the spatial dimension [Nochetto, 1993, Verfürth, 1994],
  - ✓ without the saturation assumption, in dimension 1, 2 or 3 but only for $k = 1$ [Bulle et al., 2020],

The Bank–Weiser estimator

Properties

- Efficiency ($\eta_{\mathrm{bw}} \leqslant C\eta_{\mathrm{err}} + \mathrm{h.o.t.}$):
  - ✓ global [Bank and Weiser, 1985], local [Nochetto, 1993, Verfürth, 1994].
- Reliability ($\eta_{\mathrm{err}} \leqslant C\eta_{\mathrm{bw}} + \mathrm{h.o.t.}$):
  - ✓ without restriction on k, V_T^+ or V_T^- but under a saturation assumption [Bank and Weiser, 1985],
  - ✓ without the saturation assumption but under some restrictions either on the space V^{bw} or on the spatial dimension [Nochetto, 1993, Verfürth, 1994],
  - ✓ without the saturation assumption, in dimension 1, 2 or 3 but only for $k = 1$ [Bulle et al., 2020],
  - ? still an open problem in the general case.

The Bank–Weiser estimator

Properties

- Efficiency ($\eta_{\mathrm{bw}} \leqslant C\eta_{\mathrm{err}} + \mathrm{h.o.t.}$):
  - ✓ global [Bank and Weiser, 1985], local [Nochetto, 1993, Verfürth, 1994].
- Reliability ($\eta_{\mathrm{err}} \leqslant C\eta_{\mathrm{bw}} + \mathrm{h.o.t.}$):
  - ✓ without restriction on k, V_T^+ or V_T^- but under a saturation assumption [Bank and Weiser, 1985],
  - ✓ without the saturation assumption but under some restrictions either on the space V^{bw} or on the spatial dimension [Nochetto, 1993, Verfürth, 1994],
  - ✓ without the saturation assumption, in dimension 1, 2 or 3 but only for $k = 1$ [Bulle et al., 2020],
  - ? still an open problem in the general case.
- Implementations:

The Bank–Weiser estimator

Properties

- Efficiency ($\eta_{\text{bw}} \leqslant C\eta_{\text{err}} + \text{h.o.t.}$):
  - ✓ global [Bank and Weiser, 1985], local [Nochetto, 1993, Verfürth, 1994].
- Reliability ($\eta_{\text{err}} \leqslant C\eta_{\text{bw}} + \text{h.o.t.}$):
  - ✓ without restriction on k, V_T^+ or V_T^- but under a saturation assumption [Bank and Weiser, 1985],
  - ✓ without the saturation assumption but under some restrictions either on the space V^{bw} or on the spatial dimension [Nochetto, 1993, Verfürth, 1994],
  - ✓ without the saturation assumption, in dimension 1, 2 or 3 but only for $k = 1$ [Bulle et al., 2020],
  - ? still an open problem in the general case.
- Implementations:
  - PLTMG (Fortran) [Bank and Weiser, 1985],

The Bank–Weiser estimator

Properties

- Efficiency ($\eta_{\text{bw}} \leqslant C\eta_{\text{err}} + \text{h.o.t.}$):
  - ✓ global [Bank and Weiser, 1985], local [Nochetto, 1993, Verfürth, 1994].
- Reliability ($\eta_{\text{err}} \leqslant C\eta_{\text{bw}} + \text{h.o.t.}$):
  - ✓ without restriction on k, V_T^+ or V_T^- but under a saturation assumption [Bank and Weiser, 1985],
  - ✓ without the saturation assumption but under some restrictions either on the space V^{bw} or on the spatial dimension [Nochetto, 1993, Verfürth, 1994],
  - ✓ without the saturation assumption, in dimension 1, 2 or 3 but only for $k = 1$ [Bulle et al., 2020],
  - ? still an open problem in the general case.
- Implementations:
  - PLTMG (Fortran) [Bank and Weiser, 1985],
  - IFISS (Matlab) [Liao and Silvester, 2012], [Khan et al., 2019],

The Bank–Weiser estimator

Properties

- Efficiency ($\eta_{\mathrm{bw}} \leqslant C\eta_{\mathrm{err}} + \mathrm{h.o.t.}$):
  - ✓ global [Bank and Weiser, 1985], local [Nochetto, 1993, Verfürth, 1994].
- Reliability ($\eta_{\mathrm{err}} \leqslant C\eta_{\mathrm{bw}} + \mathrm{h.o.t.}$):
  - ✓ without restriction on k, V_T^+ or V_T^- but under a saturation assumption [Bank and Weiser, 1985],
  - ✓ without the saturation assumption but under some restrictions either on the space V^{bw} or on the spatial dimension [Nochetto, 1993, Verfürth, 1994],
  - ✓ without the saturation assumption, in dimension 1, 2 or 3 but only for $k = 1$ [Bulle et al., 2020],
  - ? still an open problem in the general case.
- Implementations:
  - PLTMG (Fortran) [Bank and Weiser, 1985],
  - IFISS (Matlab) [Liao and Silvester, 2012], [Khan et al., 2019],
  - FEniCS and FEniCSx (Python, C++) [Bulle et al., 2021].

Implementation

Method details

We need to compute the matrix A_T^{bw} and vector b_T^{bw} from

$$\int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T^{\mathrm{bw}} = \int_T r_T v_T^{\mathrm{bw}} + \sum_{E \in \partial T} \frac{1}{2} \int_E J_E v_T^{\mathrm{bw}} \qquad \forall v_T^{\mathrm{bw}} \in V_T^{\mathrm{bw}}.$$

Implementation

Method details

We need to compute the matrix A_T^{bw} and vector b_T^{bw} from

$$\int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T^{\mathrm{bw}} = \int_T r_T v_T^{\mathrm{bw}} + \sum_{E \in \partial T} \frac{1}{2} \int_E J_E v_T^{\mathrm{bw}} \qquad \forall v_T^{\mathrm{bw}} \in V_T^{\mathrm{bw}}.$$

Problem: the space V_T^{bw} is not provided by DOLFIN(x).

Implementation

Method details

We need to compute the matrix A_T^{bw} and vector b_T^{bw} from

$$\int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T^{\mathrm{bw}} = \int_T r_T v_T^{\mathrm{bw}} + \sum_{E \in \partial T} \frac{1}{2} \int_E J_E v_T^{\mathrm{bw}} \qquad \forall v_T^{\mathrm{bw}} \in V_T^{\mathrm{bw}}.$$

Problem: the space V_T^{bw} is not provided by DOLFIN(x).
Idea: we rely on the matrix A_T^+ and vector b_T^+ from

$$\int_T \nabla e_T^+ \cdot \nabla v_T^+ = \int_T r_T v_T^+ + \sum_{E \in \partial T} \frac{1}{2} \int_E J_E v_T^+ \qquad \forall v_T^+ \in V_T^+,$$

since V_T^+ is provided by DOLFIN(x)

Implementation

Method details

We need to compute the matrix A_T^{bw} and vector b_T^{bw} from

$$\int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T^{\mathrm{bw}} = \int_T r_T v_T^{\mathrm{bw}} + \sum_{E \in \partial T} \frac{1}{2} \int_E J_E v_T^{\mathrm{bw}} \qquad \forall v_T^{\mathrm{bw}} \in V_T^{\mathrm{bw}}.$$

Problem: the space V_T^{bw} is not provided by DOLFIN(x).
Idea: we rely on the matrix A_T^+ and vector b_T^+ from

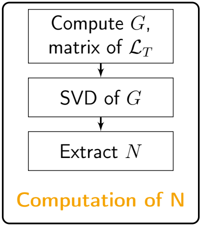
$$\int_T \nabla e_T^+ \cdot \nabla v_T^+ = \int_T r_T v_T^+ + \sum_{E \in \partial T} \frac{1}{2} \int_E J_E v_T^+ \qquad \forall v_T^+ \in V_T^+,$$

since V_T^+ is provided by DOLFIN(x) and we look for a matrix N such that:

$$A_T^{\mathrm{bw}} = N^{\mathsf{t}} A_T^+ N, \quad \text{and} \quad b_T^{\mathrm{bw}} = N^{\mathsf{t}} b_T^+.$$

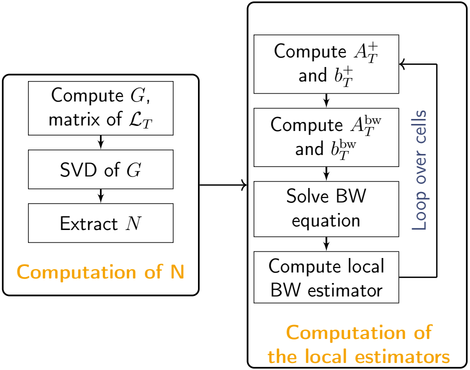
Implementation

Method details



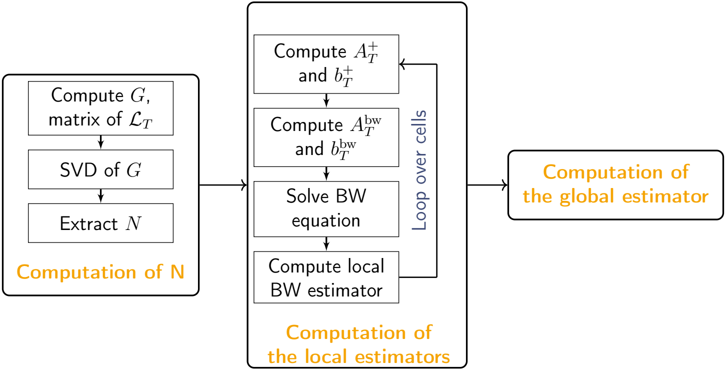
Implementation

Method details



Implementation

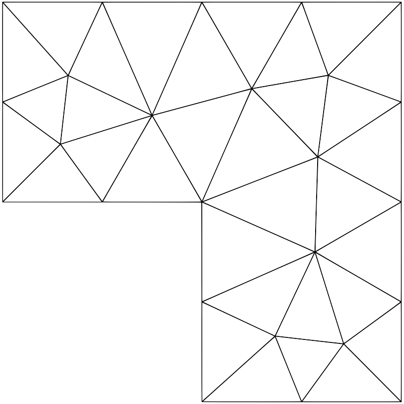
Method details



Implementation

Numerical results

Adaptive finite elements for a Poisson problem:
$-\Delta u = 0$ in Ω, $\quad u = u_D$ on Γ. Linear finite elements.

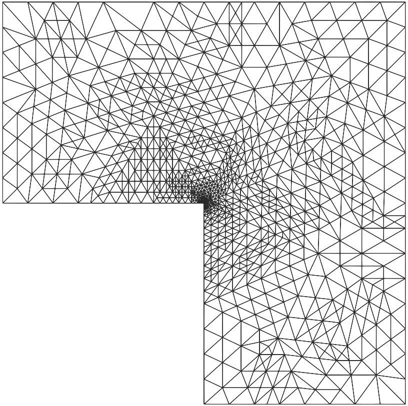


Implementation

Numerical results

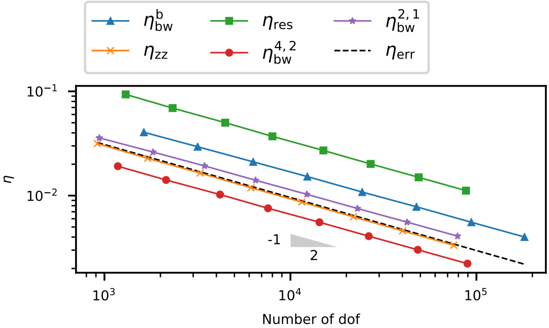
Adaptive finite elements for a Poisson problem:

$-\Delta u = 0$ in Ω, $\quad u = u_D$ on Γ. Linear finite elements.



Adaptive finite elements for a Poisson problem:

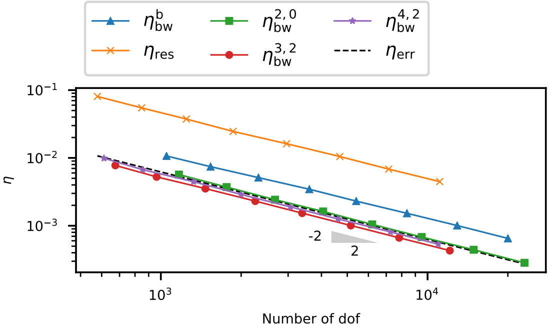
$-\Delta u = 0$ in Ω, $\quad u = u_D$ on Γ. Linear finite elements.



Notation	V_T^+	V_T^-
$\eta_{\mathrm{bw}}^{k_+,k_-}$	V_T^{k+}	V_T^{k-}
η_{bw}^{b}	V_T^2 + bubble	V_T^1

Adaptive finite elements for a Poisson problem:

$-\Delta u = 0$ in Ω, $\quad u = u_D$ on Γ. Quadratic finite elements.

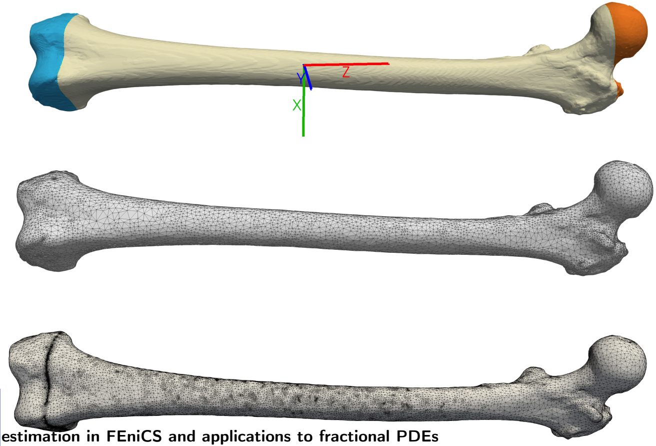


Notation	V_T^+	V_T^-
$\eta_{bw}^{k_+,k_-}$	$V_T^{k_+}$	$V_T^{k_-}$
η_{bw}^{b}	$V_T^2 + \text{bubble}$	V_T^1

Implementation

Numerical results

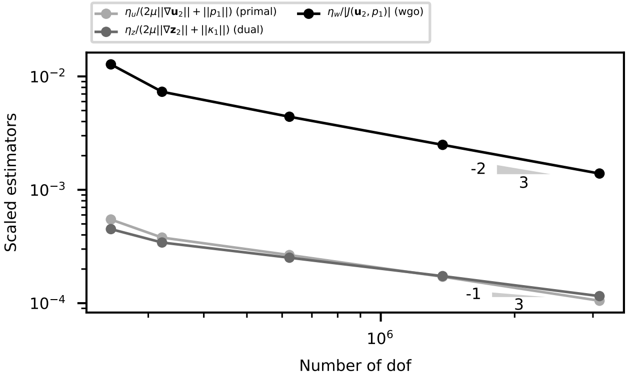
GO AFEM for a linear elasticity problem:
we used a technique from [Khan et al., 2019] to compute the estimators.
The goal functional is defined by $J(\mathbf{u_2}, p_1) := \int_\Gamma \mathbf{u_2} \cdot \mathbf{n} c$.



Implementation

Numerical results

GO AFEM for a linear elasticity problem:
we used a technique from [Khan et al., 2019] to compute the estimators.
The goal functional is defined by $J(\mathbf{u_2}, p_1) := \int_\Gamma \mathbf{u_2} \cdot \mathbf{n} c$.

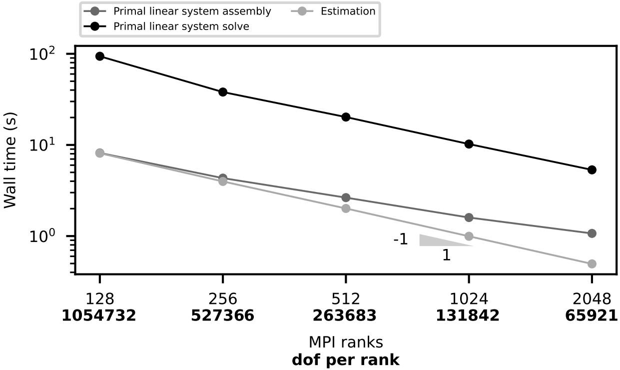


Implementation

Numerical results

Timescale study:

strong scaling study of the DOLFINx version on the Uni Lu cluster.
$-\Delta u = f$ on $[0,1]^3$, $u = 0$ on Γ. $\mathcal{P}_2$ Lagrange elements.
The Bank–Weiser estimator is $\eta_{\mathrm{bw}}^{3,2}$.
The problem size is fixed around 135 million dof.



The spectral fractional Laplacian

Problem setting

Fractional operators are used in a wide range of different fields such as statistics, hydrogeology, finance, physics...

The spectral fractional Laplacian

Problem setting

Fractional operators are used in a wide range of different fields such as statistics, hydrogeology, finance, physics...

- Main advantage: they are nonlocal.

The spectral fractional Laplacian

Problem setting

Fractional operators are used in a wide range of different fields such as statistics, hydrogeology, finance, physics...

- Main advantage: they are nonlocal.
- Main drawback: they are nonlocal.

The spectral fractional Laplacian

Problem setting

Let $\Omega \subset \mathbb{R}^d$, $s \in (0,1)$ and $f \in L^2(\Omega)$.

$$(-\Delta)^s u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma.$$

The spectral fractional Laplacian

Problem setting

Let $\Omega \subset \mathbb{R}^d$, $s \in (0,1)$ and $f \in L^2(\Omega)$.

$$(-\Delta)^s u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma.$$

Let $\{\psi_i, \lambda_i\}_{i=1}^{+\infty} \subset L^2(\Omega) \times \mathbb{R}^+$ be such that

$$-\Delta \psi_i = \lambda_i \psi_i \quad \text{in } \Omega, \qquad \psi_i = 0 \quad \text{on } \Gamma, \quad \forall i = [\![1, +\infty[\![.$$

The spectral fractional Laplacian

Problem setting

Let $\Omega \subset \mathbb{R}^d$, $s \in (0,1)$ and $f \in L^2(\Omega)$.

$$(-\Delta)^s u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma.$$

Let $\{\psi_i, \lambda_i\}_{i=1}^{+\infty} \subset L^2(\Omega) \times \mathbb{R}^+$ be such that

$$-\Delta \psi_i = \lambda_i \psi_i \quad \text{in } \Omega, \qquad \psi_i = 0 \quad \text{on } \Gamma, \quad \forall i = [\![1, +\infty[\![.$$

The solution u is defined by

$$u := \sum_{i=1}^{+\infty} \lambda_i^{-s} \left(f, \psi_i\right)_{L^2}.$$

The spectral fractional Laplacian

Problem setting

The natural Sobolev space associated with this problem is

$$\mathbb{H}^s(\Omega) := \left\{ v \in L^2(\Omega),\ \sum_{i=1}^{+\infty} \lambda_i^s \left(v, \psi_i\right)_{L^2}^2 < +\infty \right\},$$

of natural norm

$$\|v\|_{\mathbb{H}^s}^2 := \sum_{i=1}^{+\infty} \lambda_i^s \left(v, \psi_i\right)_{L^2}^2 .$$

The spectral fractional Laplacian

Discretization

$$(-\Delta)^s u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

How to solve this equation numerically ?

The spectral fractional Laplacian

Discretization

$$(-\Delta)^s u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

How to solve this equation numerically ?

Considering

$$u := (-\Delta)^{-s} f = \sum_{i=1}^{+\infty} \lambda_i^{-s} (f, \psi_i)_{L^2},$$

we use a rational approximation

$$\lambda^{-s} \simeq \mathcal{Q}_s^N(\lambda) := C_s(N) \sum_{l=1}^{N} a_l (1 + b_l \lambda)^{-1}, \qquad \forall \lambda \in [\lambda_1, +\infty),$$

where $(a_l)_l$ and $(b_l)_l$ are positive coefficients and $C_s(N)$ is independent of λ.

The spectral fractional Laplacian

Discretization

$$u = (-\Delta)^{-s} f = \sum_{i=1}^{+\infty} \lambda_i^{-s} \left(f, \psi_i\right)_{L^2}$$

The spectral fractional Laplacian

Discretization

$$
\begin{aligned}
u = (-\Delta)^{-s} f &= \sum_{i=1}^{+\infty} \lambda_i^{-s} \, (f, \psi_i)_{L^2} \\
&\simeq \sum_{i=1}^{+\infty} \mathcal{Q}_s^N(\lambda_i) \, (f, \psi_i)_{L^2}
\end{aligned}
$$

The spectral fractional Laplacian

Discretization

$$
\begin{aligned}
u = (-\Delta)^{-s} f &= \sum_{i=1}^{+\infty} \lambda_i^{-s} \, (f, \psi_i)_{L^2} \\
&\simeq \sum_{i=1}^{+\infty} \mathcal{Q}_s^N(\lambda_i) \, (f, \psi_i)_{L^2} \\
&\simeq \sum_{i=1}^{+\infty} C_s(N) \sum_{l=1}^{N} a_l (1 + b_l \lambda_i)^{-1} \, (f, \psi_i)_{L^2}
\end{aligned}
$$

The spectral fractional Laplacian

Discretization

$$
\begin{aligned}
u = (-\Delta)^{-s} f &= \sum_{i=1}^{+\infty} \lambda_i^{-s} \, (f, \psi_i)_{L^2} \\
&\simeq \sum_{i=1}^{+\infty} \mathcal{Q}_s^N(\lambda_i) \, (f, \psi_i)_{L^2} \\
&\simeq \sum_{i=1}^{+\infty} C_s(N) \sum_{l=1}^{N} a_l (1 + b_l \lambda_i)^{-1} \, (f, \psi_i)_{L^2} \\
&\simeq C_s(N) \sum_{l=1}^{N} a_l \sum_{i=1}^{+\infty} (1 + b_l \lambda_i)^{-1} \, (f, \psi_i)_{L^2}
\end{aligned}
$$

The spectral fractional Laplacian

Discretization

$$
\begin{aligned}
u = (-\Delta)^{-s} f &= \sum_{i=1}^{+\infty} \lambda_i^{-s} \, (f, \psi_i)_{L^2} \\
&\simeq \sum_{i=1}^{+\infty} \mathcal{Q}_s^N(\lambda_i) \, (f, \psi_i)_{L^2} \\
&\simeq \sum_{i=1}^{+\infty} C_s(N) \sum_{l=1}^{N} a_l (1 + b_l \lambda_i)^{-1} \, (f, \psi_i)_{L^2} \\
&\simeq C_s(N) \sum_{l=1}^{N} a_l \sum_{i=1}^{+\infty} (1 + b_l \lambda_i)^{-1} \, (f, \psi_i)_{L^2} \\
&\simeq C_s(N) \sum_{l=1}^{N} a_l (\mathrm{Id} - b_l \Delta)^{-1} f = C_s(N) \sum_{l=1}^{N} a_l u_l.
\end{aligned}
$$

The spectral fractional Laplacian

Discretization

$$u = (-\Delta)^{-s} f \quad \longrightarrow \quad u_l = (\mathrm{Id} - b_l \Delta)^{-1} f, \quad \forall l \in [1, N].$$

The spectral fractional Laplacian

Discretization

$$u = (-\Delta)^{-s} f \quad \longrightarrow \quad u_l = (\mathrm{Id} - b_l \Delta)^{-1} f, \quad \forall l \in [1, N].$$

We denote

$$u \simeq u^N := C_s(N) \sum_{l=1}^{N} a_l u_l.$$

However, u^N is not a discrete function.

The spectral fractional Laplacian

Discretization

$$u = (-\Delta)^{-s} f \quad \longrightarrow \quad u_l = (\mathrm{Id} - b_l \Delta)^{-1} f, \quad \forall l \in [1, N].$$

We denote

$$u \simeq u^N := C_s(N) \sum_{l=1}^{N} a_l u_l.$$

However, u^N is not a discrete function. To get a full discretization, we use a FE method. We reformulate the problems in the weak form

$$\int_\Omega u_l v + b_l \int_\Omega \nabla u_l \cdot \nabla v = \int_\Omega f v, \qquad \forall v \in H_0^1(\Omega),\ \forall l \in [1, N],$$

The spectral fractional Laplacian

Discretization

$$u = (-\Delta)^{-s} f \quad \longrightarrow \quad u_l = (\mathrm{Id} - b_l \Delta)^{-1} f, \quad \forall l \in [1, N].$$

We denote

$$u \simeq u^N := C_s(N) \sum_{l=1}^{N} a_l u_l.$$

However, u^N is not a discrete function. To get a full discretization, we use a FE method. We reformulate the problems in the weak form

$$\int_\Omega u_l v + b_l \int_\Omega \nabla u_l \cdot \nabla v = \int_\Omega f v, \qquad \forall v \in H_0^1(\Omega),\ \forall l \in [1, N],$$

and write its FE discretization

$$\int_\Omega u_{l,k} v_k + b_l \int_\Omega \nabla u_{l,k} \cdot \nabla v_k = \int_\Omega f v_k, \qquad \forall v_k \in V^1,\ \forall l \in [1, N].$$

The spectral fractional Laplacian

Discretization

Solving these classical FE problems we finally get a fully discrete approximation of u

$$u \simeq u^N := C_s(N) \sum_{l=1}^{N} a_l u_l \simeq C_s(N) \sum_{l=1}^{N} a_l u_{l,k} =: u_k^N$$

The spectral fractional Laplacian

Discretization

Solving these classical FE problems we finally get a fully discrete approximation of u

$$u \simeq u^N := C_s(N) \sum_{l=1}^{N} a_l u_l \simeq C_s(N) \sum_{l=1}^{N} a_l u_{l,k} =: u_k^N$$

The main advantages of this kind of methods is that they are easily parallelizable and involve "standard" FE machinery.

A posteriori error estimation

Rational approximation error

The next question is:
how can we bound the discretization error ?

$$\mathrm{err} := \|u - u_k^N\| \leqslant \|u - u^N\| + \|u^N - u_k^N\|.$$

A posteriori error estimation

Rational approximation error

The next question is:
how can we bound the discretization error ?

$$\mathrm{err} := \|u - u_k^N\| \leqslant \|u - u^N\| + \|u^N - u_k^N\|.$$

Two sources of error:

- the rational approximation error $\|u - u^N\|$,
- the finite element error $\|u^N - u_k^N\|$.

where $\|\cdot\| = \|\cdot\|_{L^2}$, or $\|\cdot\|_{\mathbb{H}^s}$.

A posteriori error estimation

Rational approximation error

If there exists $\varepsilon(N) \underset{N \to +\infty}{\longrightarrow} 0$ such that

$$\left|\lambda^{-s} - \mathcal{Q}_s^N(\lambda)\right| \leqslant \varepsilon(N), \qquad \forall \lambda \in [\lambda_1, +\infty),$$

then, [Bonito and Pasciak, 2015]

$$\|u - u^N\|_{L^2} \leqslant \varepsilon(N) \|f\|_{L^2}.$$

Moreover, if $f \in \mathbb{H}^s(\Omega)$, then [Bonito and Pasciak, 2016]

$$\|u - u^N\|_{\mathbb{H}^s} \leqslant \varepsilon(N) \|f\|_{\mathbb{H}^s}.$$

A posteriori error estimation

Rational approximation error

If there exists $\varepsilon(N) \underset{N\to+\infty}{\longrightarrow} 0$ such that

$$\left|\lambda^{-s} - \mathcal{Q}_s^N(\lambda)\right| \leqslant \varepsilon(N), \qquad \forall \lambda \in [\lambda_1, +\infty),$$

then, [Bonito and Pasciak, 2015]

$$\|u - u^N\|_{L^2} \leqslant \varepsilon(N)\|f\|_{L^2}.$$

Moreover, if $f \in \mathbb{H}^s(\Omega)$, then [Bonito and Pasciak, 2016]

$$\|u - u^N\|_{\mathbb{H}^s} \leqslant \varepsilon(N)\|f\|_{\mathbb{H}^s}.$$

In particular, there exists an approximation $\mathcal{Q}_s^N$ such that [Bonito and Pasciak, 2015]

$$\varepsilon(N) = \underset{N\to+\infty}{\mathcal{O}}\left(\mathrm{e}^{-\left(\pi^2/2\sqrt{2}\right)\sqrt{N}}\right).$$

A posteriori error estimation

Rational approximation error

Conjecture: $\|u - u^N\|_{\mathbb{H}^s} \leqslant \varepsilon(N)\|f\|_{L^2}.$

A posteriori error estimation

Rational approximation error

Conjecture: $\|u - u^N\|_{\mathbb{H}^s} \leqslant \varepsilon(N)\|f\|_{L^2}$.
What we can prove currently (not published yet):

$$\|u - u^N\|_{\mathbb{H}^s} \leqslant \widetilde{\varepsilon}(N)\|f\|_{L^2},$$

where $\widetilde{\varepsilon}(N) \underset{N\to+\infty}{\longrightarrow} 0$ with a possibly slower convergence rate than ε.

A posteriori error estimation

Finite element error

What about $\|u^N - u_k^N\|$?

A posteriori error estimation

Finite element error

What about $\|u^N - u_k^N\|$?

A priori error estimates in [Bonito and Pasciak, 2015] and [Bonito and Pasciak, 2016].

A posteriori error estimation

Finite element error

What about $\|u^N - u_k^N\|$?

A priori error estimates in [Bonito and Pasciak, 2015] and [Bonito and Pasciak, 2016].

We are looking for a computable quantity η such that

$$\|u^N - u_k^N\| \simeq \eta.$$

A posteriori error estimation

Finite element error

Heuristics L^2 case:

$$u^N - u_k^N = C_s(N) \sum_{l=1}^{N} a_l (u_l - u_{l,k}),$$

We use the Bank–Weiser solution to quantify

$$u_l - u_{l,k} \simeq e_l^{\mathrm{bw}}.$$

A posteriori error estimation

Finite element error

Heuristics L^2 case:

$$u^N - u_k^N = C_s(N) \sum_{l=1}^{N} a_l (u_l - u_{l,k}),$$

We use the Bank–Weiser solution to quantify

$$u_l - u_{l,k} \simeq e_l^{\mathrm{bw}}.$$

Then,

$$u^N - u_k^N \simeq C_s(N) \sum_{l=1}^{N} a_l e_l^{\mathrm{bw}} =: e^{\mathrm{bw},N}.$$

A posteriori error estimation

Finite element error

Heuristics L^2 case:

$$u^N - u_k^N = C_s(N) \sum_{l=1}^{N} a_l (u_l - u_{l,k}),$$

We use the Bank–Weiser solution to quantify

$$u_l - u_{l,k} \simeq e_l^{\mathrm{bw}}.$$

Then,

$$u^N - u_k^N \simeq C_s(N) \sum_{l=1}^{N} a_l e_l^{\mathrm{bw}} =: e^{\mathrm{bw},N}.$$

Finally, we hope that:

$$\|u^N - u_k^N\|_{L^2} \simeq \|e^{\mathrm{bw},N}\|_{L^2}.$$

A posteriori error estimation

Finite element error

Heuristics $\mathbb{H}^s$ case:

$$u^N - u_k^N = C_s(N) \sum_{l=1}^{N} a_l (u_l - u_{l,k}),$$

We use the Bank–Weiser solution to quantify

$$|||u_l - u_{l,k}|||_l \simeq |||e_l^{\mathrm{bw}}|||_l,$$

where $|||v|||_l^2 := \|v\|_{L^2}^2 + b_l |v|_{H^1}^2$.

A posteriori error estimation

Finite element error

Heuristics $\mathbb{H}^s$ case:

$$u^N - u_k^N = C_s(N) \sum_{l=1}^{N} a_l (u_l - u_{l,k}),$$

We use the Bank–Weiser solution to quantify

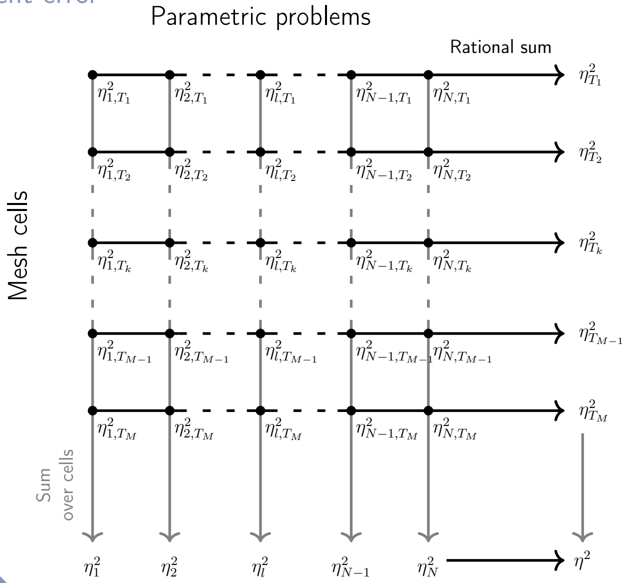
$$|||u_l - u_{l,k}|||_l \simeq |||e_l^{\mathrm{bw}}|||_l,$$

where $|||v|||_l^2 := \|v\|_{L^2}^2 + b_l |v|_{H^1}^2$. Then, we hope that:

$$\|u^N - u_k^N\|_{\mathbb{H}^s}^2 \simeq C_s(N) \sum_{l=1}^{N} a_l |||u_l - u_{l,k}|||_l^2 \simeq C_s(N) \sum_{l=1}^{N} a_l |||e_l^{\mathrm{bw}}|||_l^2.$$

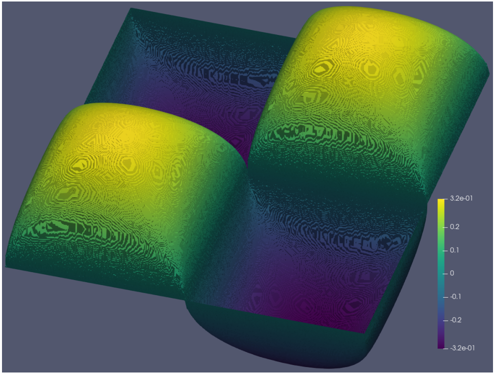
A posteriori error estimation

Finite element error



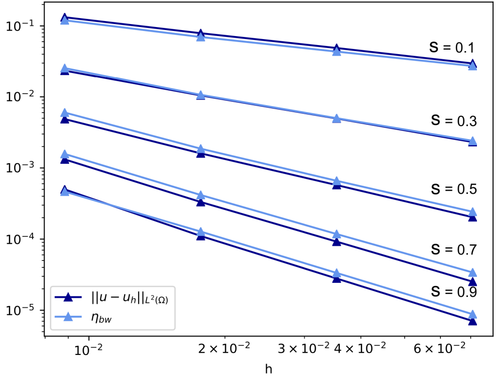
Numerical results

$(-\Delta)^s u = f$, in $[0,1]^2$, $u = 0$, on Γ,
with $f(x,y) = 1$ in $[0,0.5]^2 \cup [0.5,1]^2$, -1 otherwise.
We assume the rational approximation is negligible, i.e. $u = u^N$.



Numerical results

Uniform mesh refinement.

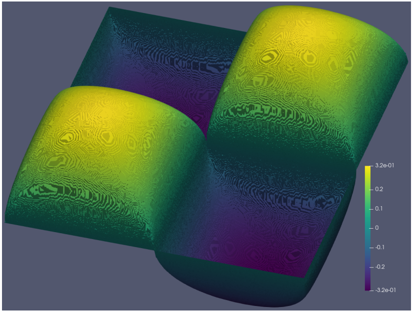


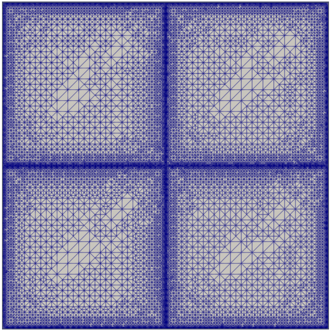
s	0.1	0.3	0.5	0.7	0.9
Th. slope	0.7	1.1	1.5	1.9	2.0
Err. slope	0.71	1.11	1.52	1.9	2.04
Est. slope	0.71	1.13	1.54	1.84	1.91

[Bonito and Pasciak, 2015]

Numerical results

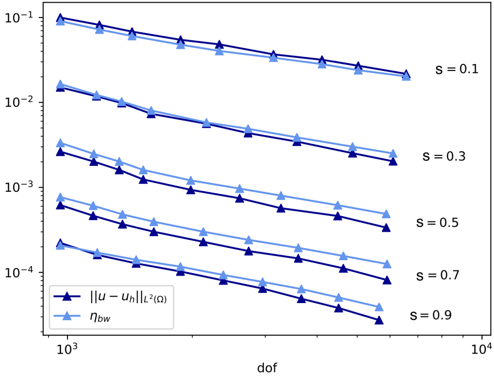
Adaptive mesh refinement.





Numerical results

Adaptive mesh refinement.



s	0.1	0.3	0.5	0.7	0.9
Th. slope (unif.)	0.35	0.55	0.75	0.95	1.0
Err. slope (adapt.)	0.71	0.81	0.86	0.88	0.99
Est. slope (adapt.)	0.72	0.79	0.85	0.89	0.96

[Bonito and Pasciak, 2015]

References I

Bank, R. E. and Weiser, A. (1985).
Some A Posteriori Error Estimators for Elliptic Partial Differential Equations.
Math. Comput., 44(170):283–301.

Bonito, A. and Pasciak, J. E. (2015).
Numerical approximation of fractional powers of elliptic operators.
Math. Comput., (295).

Bonito, A. and Pasciak, J. E. (2016).
Numerical approximation of fractional powers of regularly accretive operators.
IMA J. Numer. Anal., 37(3):drw042.

Bulle, R., Chouly, F., Hale, J. S., and Lozinski, A. (2020).
Removing the saturation assumption in Bank–Weiser error estimator analysis in dimension three.
Appl. Math. Lett., 107(1):106429.

Bulle, R., Hale, J. S., Lozinski, A., Bordas, S. P. A., and Chouly, F. (2021).
Hierarchical a posteriori error estimation of Bank-Weiser type in the FEniCS Project.

References II

Khan, A., Powell, C. E., and Silvester, D. J. (2019).
Robust a posteriori error estimators for mixed approximation of nearly incompressible elasticity.
Int. J. Numer. Methods Eng., 119(1):18–37.

Liao, Q. and Silvester, D. (2012).
A simple yet effective a posteriori estimator for classical mixed approximation of Stokes equations.
Appl. Numer. Math., 62(9):1242–1256.

Nochetto, R. H. (1993).
Removing the saturation assumption in a posteriori error analysis.
Istit. Lomb. Accad. Sci. Lett. Rend. A, 127(1):67—-82 (1994).

Verfürth, R. (1994).
A posteriori error estimation and adaptive mesh-refinement techniques.
J. Comput. Appl. Math., 50(1-3):67–83.

Thank you for your attention!

I would like to acknowledge the support of the ASSIST research project of the University of Luxembourg. This presentation has been prepared in the framework of the DRIVEN project funded by the European Union's Horizon 2020 Research and Innovation programme under Grant Agreement No. 811099.