Hierarchical A Posteriori Error Estimation of Bank–Weiser Type in the FEniCS Project

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    - Completing the work of [?].

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A posteriori error estimation

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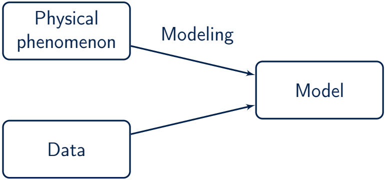
A posteriori error estimation



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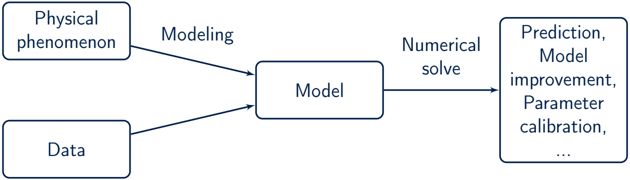

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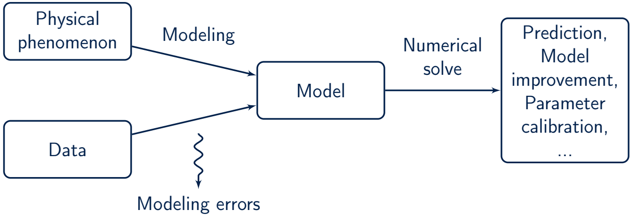

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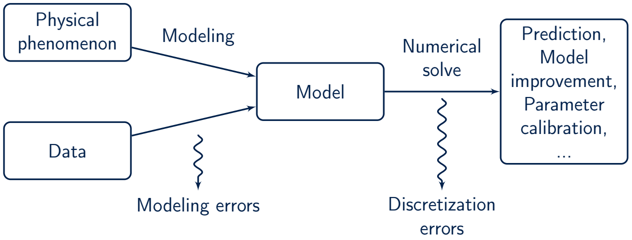

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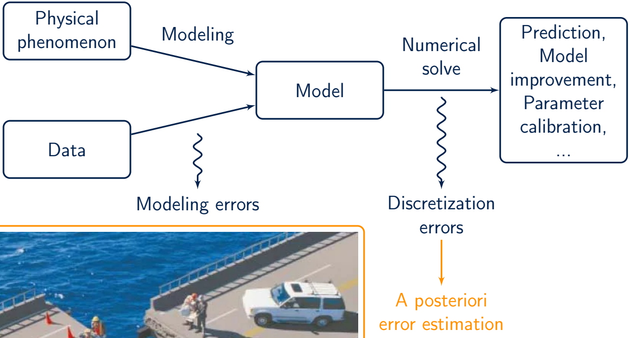

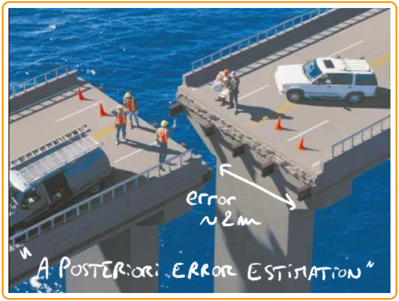

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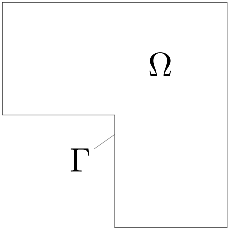
Toy problem setting



Let $f \in L^2(\Omega)$, we look for u with sufficient regularity s.t.

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma.$$

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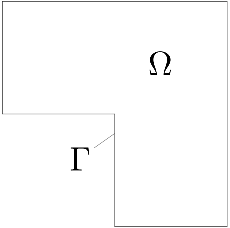

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In weak formulation, find u in $H^1_0(\Omega)$ such that

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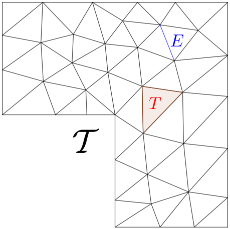

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Lagrange finite element discretization of order k, find u_k in V^k such that

$$\int_\Omega \nabla u_k \cdot \nabla v_k = \int_\Omega f v_k \quad \forall v_k \in V^k.$$

Definition of the Bank–Weiser estimator

Toy problem setting

We quantify the discretization error $e := u_k - u$ using the energy norm $\eta_{\mathrm{err}} := \|\nabla e\|_\Omega = \|\nabla u_k - \nabla u\|_\Omega$.

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Toy problem setting

We quantify the discretization error $e := u_k - u$ using the energy norm $\eta_{\mathrm{err}} := \|\nabla e\|_\Omega = \|\nabla u_k - \nabla u\|_\Omega$.

Goal: estimate η i.e. find a computable quantity η_{bw} such that

$$\eta_{\mathrm{bw}} \approx \eta_{\mathrm{err}}.$$

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The Bank–Weiser estimator

The restriction e_T of e to any cell T of the mesh satisfies the equation

$$\int_T \nabla e_T \cdot \nabla v_T := \int_T (f - \Delta u_k) v_T + \sum_{E \in \partial T} \frac{1}{2} \int_E [\![\partial_n u_k]\!]_E \, v_T \quad \forall v \in H^1_0(T).$$

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On a cell T, the Bank–Weiser problem is given by:
find e_T^{bw} in V_T^{bw} such that

$$\int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T^{\mathrm{bw}} = \int_T (f - \Delta u_k) v_T^{\mathrm{bw}} + \sum_{E \in \partial T} \frac{1}{2} \int_E [\![\partial_n u_k]\!]_E \, v_T^{\mathrm{bw}} \quad \forall v_T^{\mathrm{bw}} \in V_T^{\mathrm{bw}}.$$

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The Bank–Weiser estimator is defined as

$$\eta_{\mathrm{bw}}^2 := \sum_{T \in \mathcal{T}} \eta_{\mathrm{bw},T}^2, \quad \eta_{\mathrm{bw},T} := \|\nabla e_T^{\mathrm{bw}}\|_T.$$

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The Bank–Weiser estimator

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How is V_T^{bw} defined ?

- Different definitions of V_T^{bw} lead to different variants of the Bank–Weiser estimator.
- General principle: let $V_T^- \subsetneq V_T^+$ be two finite element spaces and

$$\mathcal{L}_T : V_T^+ \longrightarrow V_T^-,$$

be the local Lagrange interpolation operator,

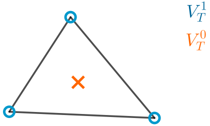
$$V_T^{\mathrm{bw}} := \ker(\mathcal{L}_T) = \left\{ v_T^+ \in V_T^+,\ \mathcal{L}_T(v_T^+) = 0 \right\}.$$

Definition of the Bank–Weiser estimator

The Bank–Weiser estimator

Examples:

- For $V_T^+ = V_T^1$ and $V_T^- = V_T^0$:



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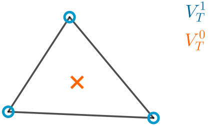

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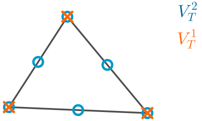

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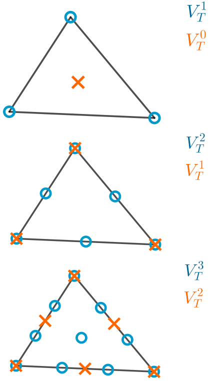

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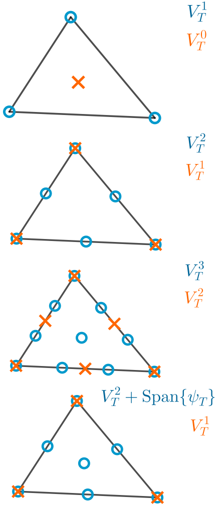
- For $V_T^+ = V_T^3$ and $V_T^- = V_T^2$:
- For $V_T^+ = V_T^2 + \mathrm{Span}\{\psi_T\}$ and $V_T^- = V_T^1$:



Definition of the Bank–Weiser estimator

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- Efficiency ($\eta_{\mathrm{bw}} \leqslant C\eta_{\mathrm{err}} + \mathrm{h.o.t.}$):

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  - ✓ without the saturation assumption but only if $\psi_T V_T^k \subset V_T^{\mathrm{bw}}$ [?],
  - ✓ without the saturation assumption, in dimension 1, 2 or 3 but only for $k = 1$ [?],
  - ? still an open problem in the general case (e.g. for $k = 2$, $V_T^+ = V_T^3$ and $V_T^- = V_T^2$).

Definition of the Bank–Weiser estimator

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  - FEniCS (Python, C++) [?].

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Implementation

Method details

We need to compute the matrix A_T^{bw} and vector b_T^{bw} from

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Problem: the space V_T^{bw} is not provided by DOLFIN.

Implementation

Method details

We need to compute the matrix A_T^{bw} and vector b_T^{bw} from

$$\int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T^{\mathrm{bw}} = \int_T (f - \Delta u_k) v_T^{\mathrm{bw}} + \sum_{E \in \partial T} \frac{1}{2} \int_E [\![\partial_n u_k]\!]_E \, v_T^{\mathrm{bw}} \quad \forall v_T^{\mathrm{bw}} \in V_T^{\mathrm{bw}}.$$

Problem: the space V_T^{bw} is not provided by DOLFIN.
Idea: we rely on the matrix A_T^{+} and vector b_T^{+} from

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since V_T^{+} is provided by DOLFIN

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Problem: the space V_T^{bw} is not provided by DOLFIN.
Idea: we rely on the matrix A_T^+ and vector b_T^+ from

$$\int_T \nabla e_T^+ \cdot \nabla v_T^+ = \int_T (f - \Delta u_k) v_T^+ + \sum_{E \in \partial T} \frac{1}{2} \int_E [\![\partial_n u_k]\!]_E \, v_T^+ \quad \forall v_T^+ \in V_T^+,$$

since V_T^+ is provided by DOLFIN and we look for a matrix N such that:

$$A_T^{\mathrm{bw}} = N^{\mathsf{t}} A_T^+ N, \quad \text{and} \quad b_T^{\mathrm{bw}} = N^{\mathsf{t}} b_T^+.$$

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Method details

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We can write:

$$V = (\xi_1^0 | \cdots | \xi_{d_{\mathrm{bw}}}^0 | \xi_1 | \cdots | \xi_{d_-}),$$

where $\{\xi_1^0, \cdots, \xi_{d_{\mathrm{bw}}}^0\}$ is a basis for V_T^{bw}.

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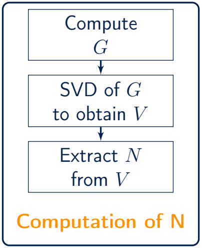
$$N = (\xi_1^0|\cdots|\xi_{d_{\mathrm{bw}}}^0).$$

Moreover, N does not depend on the cell T.

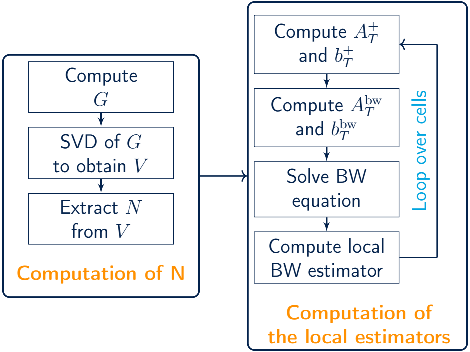
Implementation

Method details



Implementation

Method details



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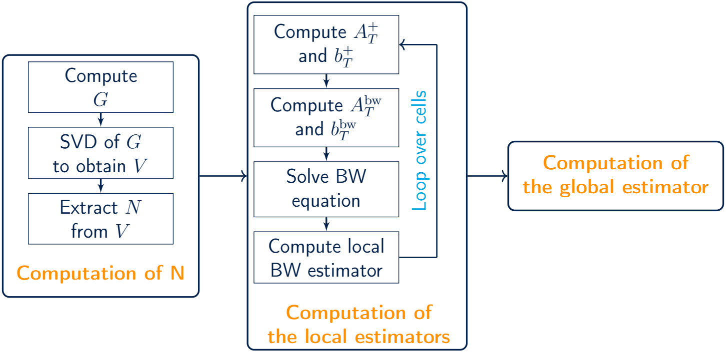

Implementation

Method details

```
def estimate(u_h):
    mesh = u_h.function_space().mesh()
    element_f = FiniteElement("DG", triangle, 2)
    element_g = FiniteElement("DG", triangle, 1)

    N = fenics_error_estimation.create_interpolation(element_f, element_g)

    V_f = FunctionSpace(mesh, element_f)
    e = TrialFunction(V_f)
    v = TestFunction(V_f)
    f = Constant(0.0)
    bcs = DirichletBC(V_f, Constant(0.0), "on_boundary", "geometric")

    n = FacetNormal(mesh)
    a_e = inner(grad(e), grad(v))*dx
    L_e = inner(f + div(grad(u_h)), v)*dx + \
          inner(jump(grad(u_h), -n), avg(v))*dS

    e_h = fenics_error_estimation.estimate(a_e, L_e, N, bcs)
    error = norm(e_h, "H10")

    V_e = FunctionSpace(mesh, "DG", 0)
    v = TestFunction(V_e)
    eta_h = Function(V_e, name="eta_h")
    eta = assemble(inner(inner(grad(e_h), grad(e_h)), v)*dx)
    eta_h.vector()[:] = eta

    return eta_h
```

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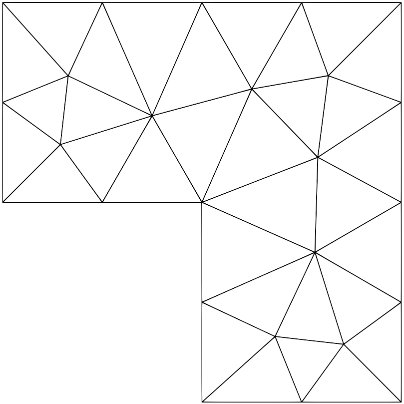
Numercial results

Adaptive finite elements for a Poisson problem:

$-\Delta u = 0$ in Ω, $\quad u = u_D$ on Γ. Linear finite elements.



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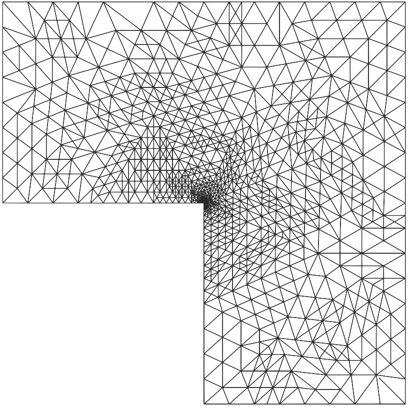

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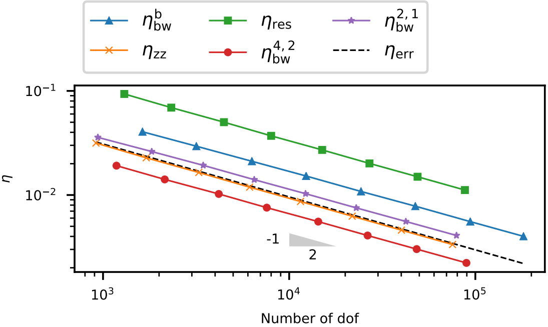

Notation	V_T^+	V_T^-
$\eta_{\mathrm{bw}}^{k_+,k_-}$	$V_T^{k_+}$	$V_T^{k_-}$
η_{bw}^{b}	$V_T^2 + \mathrm{bubble}$	V_T^1

Implementation

Numercial results

Adaptive finite elements for a Poisson problem:

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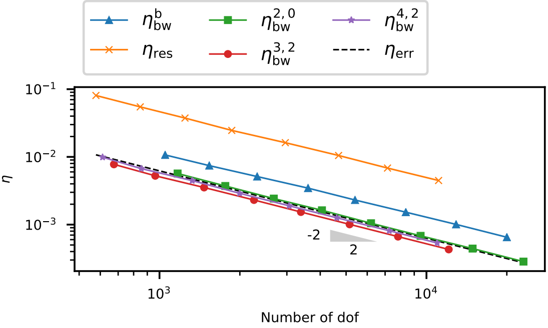

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η_{bw}^{b}	$V_T^2 + \text{bubble}$	V_T^1

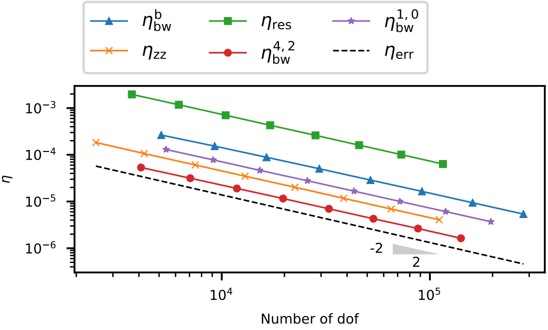
Implementation

Numercial results

Goal oriented adaptive finite elements for a Poisson problem:

$$-\Delta u = 0 \text{ in } \Omega, \quad u = u_D \text{ on } \Gamma. \qquad \eta_{\mathrm{err}} := J(u - u_1) = \int_\Omega (u - u_h)c,$$

where c is a smooth weight function.
The estimators are computed using the WGO method from [**?**].



Notation	V_T^+	V_T^-
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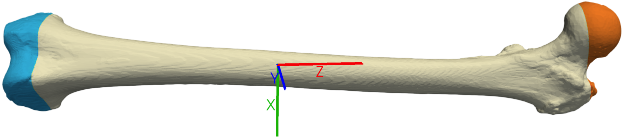
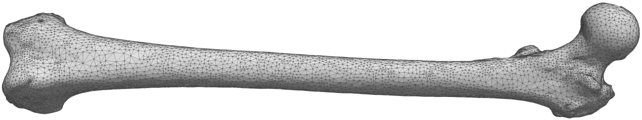
GO AFEM for a linear elasticity problem:

we used a technique from [?] to compute the estimators. The goal functional is defined by $J(\mathbf{u_2}, p_1) := \int_\Gamma \mathbf{u_2} \cdot \mathbf{n} c$.





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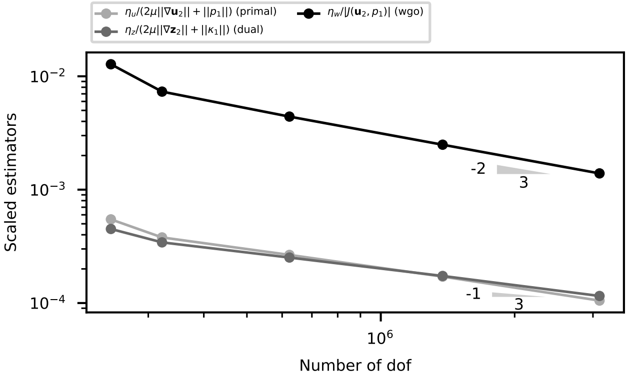

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Perspectives

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- Find a criterion to determine the best couple of spaces (V_T^+, V_T^-) for a given problem.
- Prove the reliability of Bank–Weiser estimators in the general case.
- Investigate performance of Bank–Weiser estimators for error estimation in L^2 norm.

References I

Thank you for your attention!

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