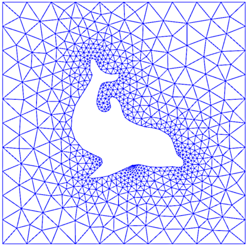
Finite Element Methods and A Posteriori Error Estimation



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- FEM: what is it for ?
- FEM: how does it work ?
- FEM: does it work ?
- A posteriori error estimation
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- Adaptive refinement process
- Variants of Bank-Weiser estimator
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FEM: what is it for ?

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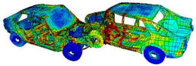
- structural mechanics



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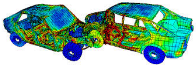
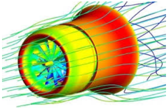
- fluid mechanics





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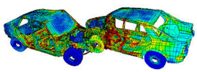
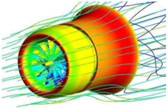
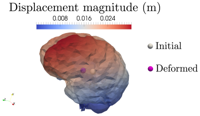
- fluid mechanics
- medicine/biomechanics







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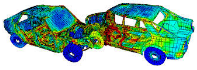
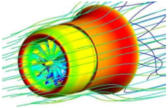
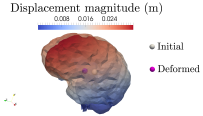
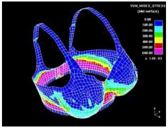
- and so on...









FEM: what is it for ?

For us, FEM will be used to solve Poisson's problem:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

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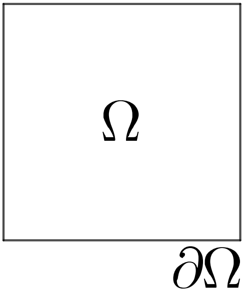

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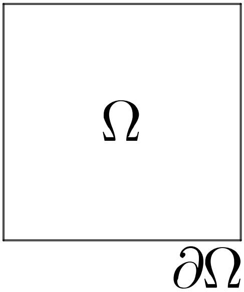

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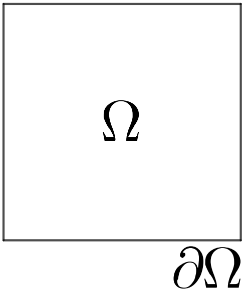

where:

- $\Omega \subset \mathbb{R}^2$,
- $f \in L^2(\Omega)$,
- $\Delta v = \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2}$.

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FEM: how does it work ?

First, we reformulate our problem:

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where,

$$H_0^1(\Omega) := \left\{ v \in L^2(\Omega), \ \nabla v \in (L^2(\Omega))^2, \ v_{|\partial\Omega} = 0 \right\}.$$

provided with the norm $\|\nabla v\|^2 = \int_\Omega \nabla v \cdot \nabla v$.

FEM: how does it work ?

Our new problem is:

Find a function u in $H_0^1(\Omega)$ such that, for any v in $H_0^1(\Omega)$,

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We want to discretize this problem in order to compute a numerical approximation u_h of u.

FEM: how does it work ?

To discretize our problem, we use finite elements:

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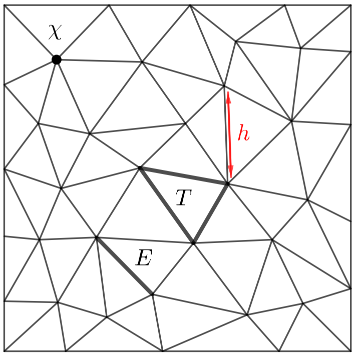

- $\mathcal{E}_h$ set of edges,
- $\mathcal{E}_h^I$ set of interior edges.

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To discretize our problem, we use finite elements:

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$$V_h^k := \left\{v_h \in \mathcal{C}^0(\Omega),\ v_{h|T} \in \mathcal{P}^k(T)\ \forall T \in \mathcal{T}_h,\ v_{h|\partial\Omega} = 0\right\}.$$

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To discretize our problem, we use finite elements:

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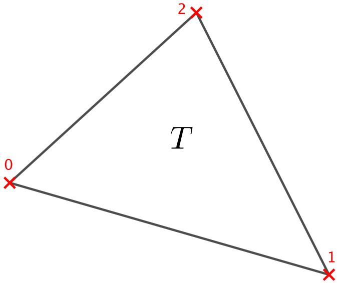
Ex: $k = 1$, v_h in V_h^1,

$$v_{h|T} = ax + by + c,$$



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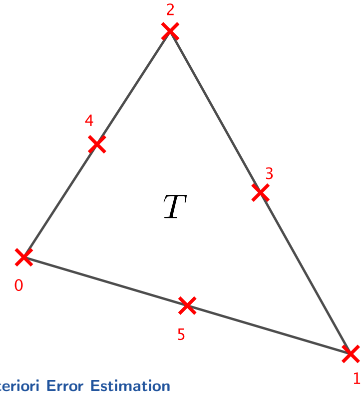
Ex: $k = 2$,

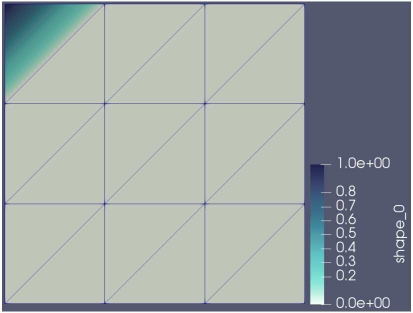


FEM: how does it work ?

To discretize our problem, we use finite elements:

$$\{\mathcal{T}_h, V_h^k, \Sigma_h\}.$$

Moreover, $V_h^k := \langle \varphi_0, \cdots, \varphi_{d_k} \rangle$, such that $\varphi_i(x_j) = \delta_{i,j}$.
Ex: $k = 1$,



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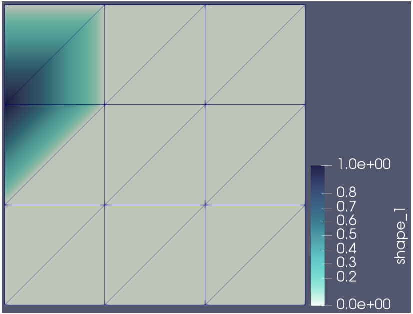

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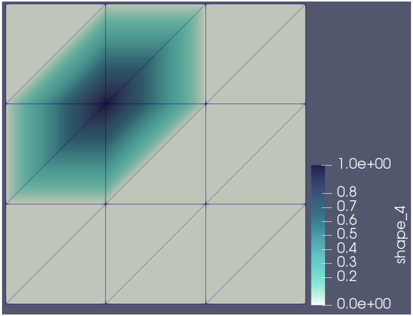

FEM: how does it work ?

Our infinite dimensional problem

Find a function u in $H_0^1(\Omega)$ such that, for any v in $H_0^1(\Omega)$,

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is discretized in

Find a function u_h in V_h^k such that, for any v_h in V_h^k,

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Let us consider a particular example,

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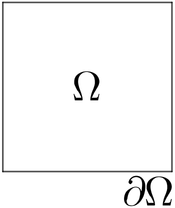

with,

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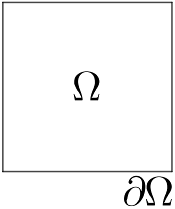

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Then,

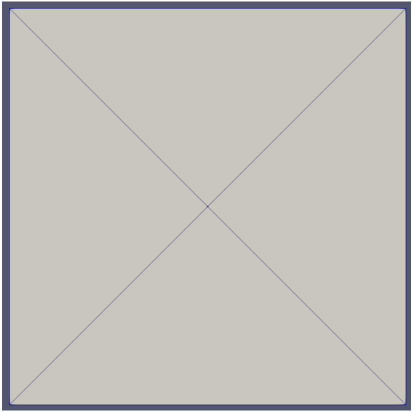
$$u(x, y) = \sin(\pi x) \sin(\pi y).$$

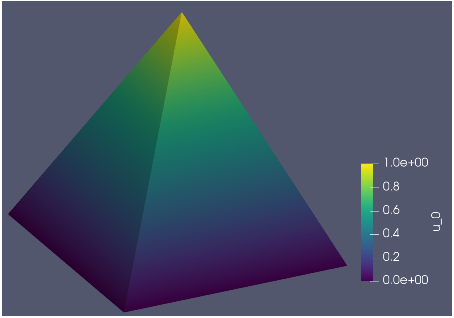
Intermission: FEniCS

All the numerical tests here have been carried out using the finite element software FEniCS. (Logg [2007], Logg and Wells [2010], Alnæs et al. [2015])

FEM: does it work ?

(k=1)





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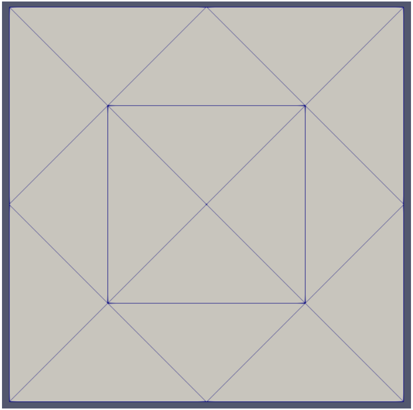

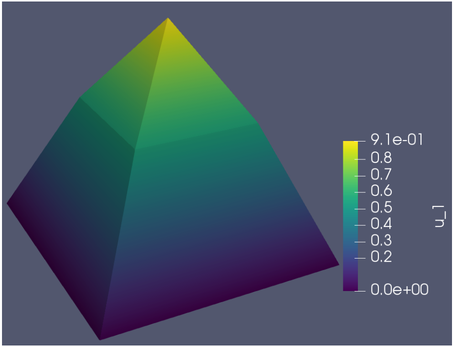

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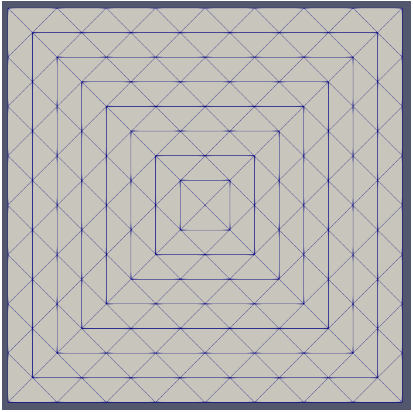

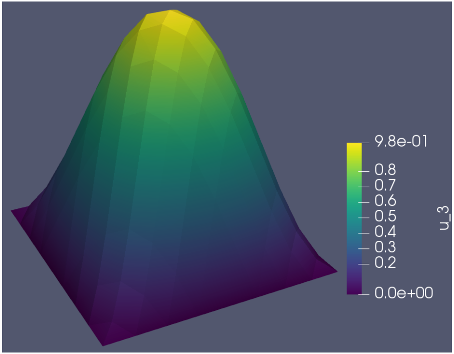

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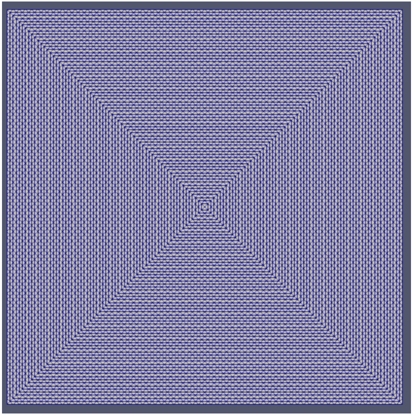

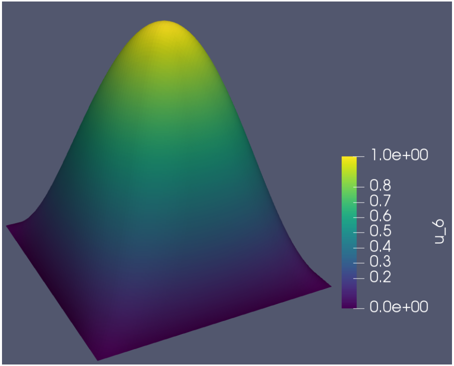

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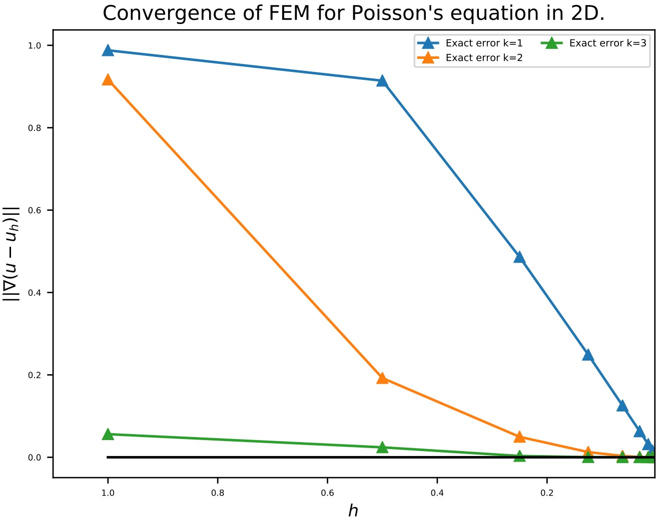

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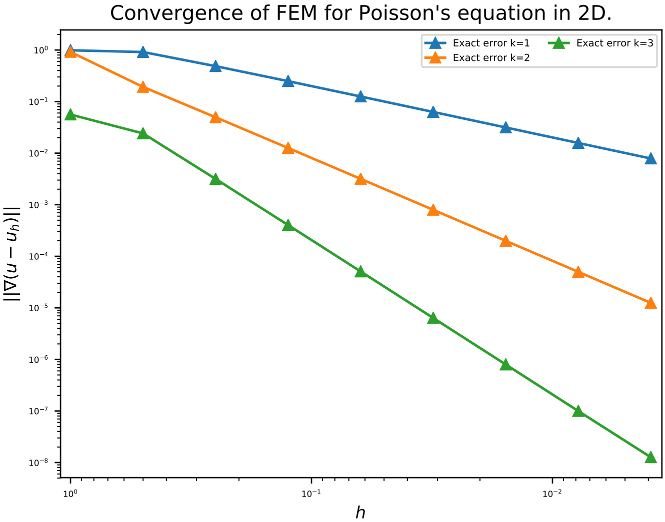

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A priori error estimation

Theorem: Let $\{\mathcal{T}_h\}_h$ be a family of meshes and $u_h \in V_h^k$ the corresponding finite element solutions of degree k, then

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Moreover, if $u \in H^{k+1}(\Omega)$ then, there exists a constant C independent of u and h such that,

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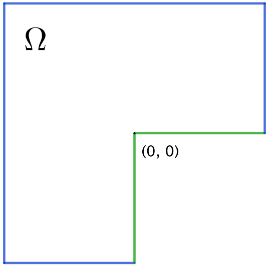
Let us check another example:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ u = u_D \neq 0 & \text{on } \Gamma_1. \end{cases}$$

We discretise this equation with FEM of degree 1.



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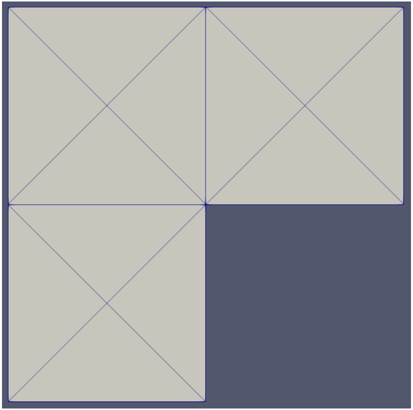

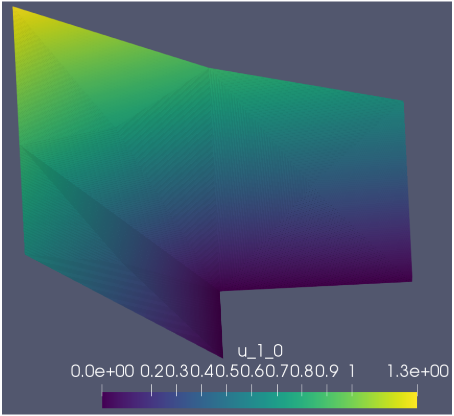

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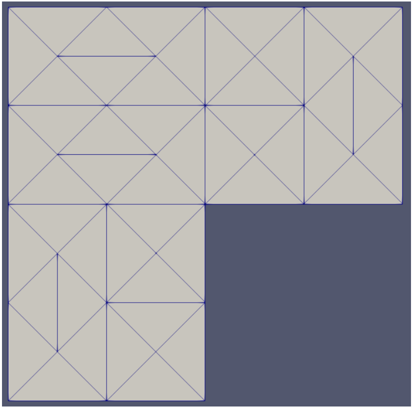

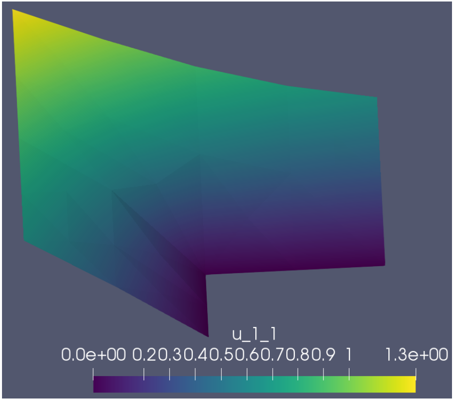

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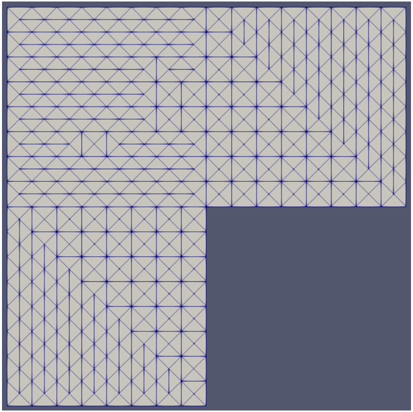

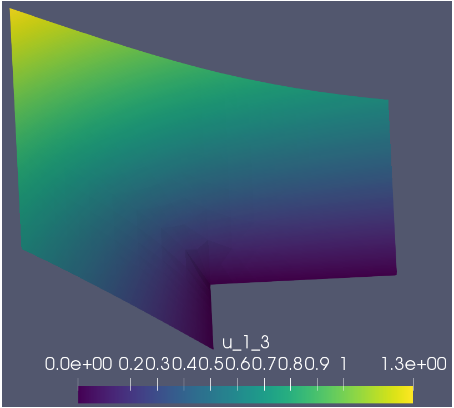

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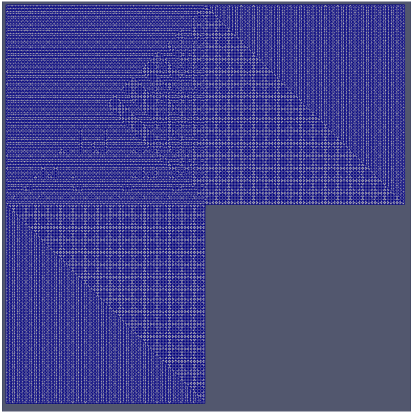

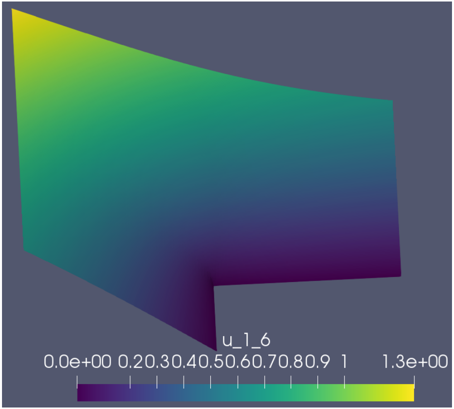

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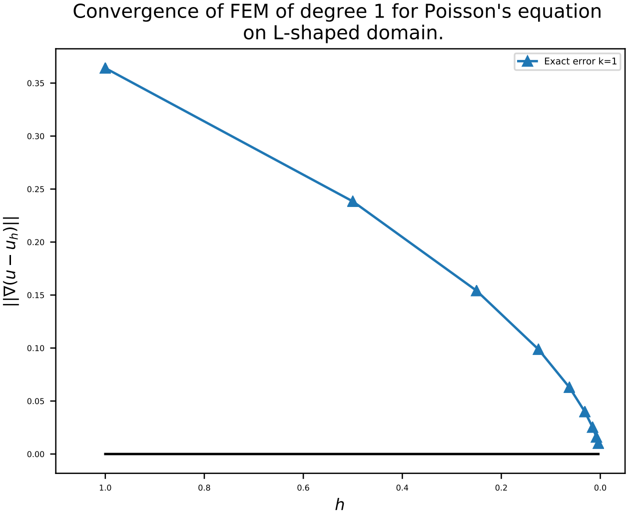

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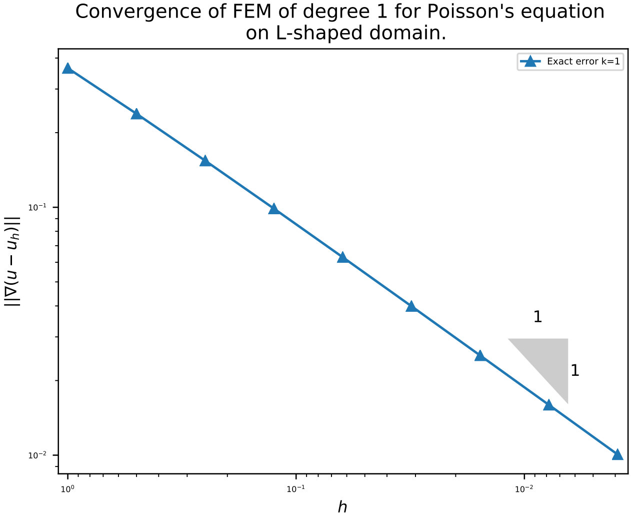

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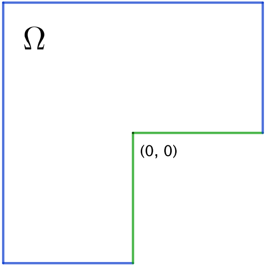

In this particular case,

$$u \in H^{1+\varepsilon}(\Omega) \quad \forall \varepsilon < \frac{2}{3},$$

especially,

$$u \notin H^2(\Omega).$$

In fact, ∇u has a singularity in $(0,0)$.

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- sometimes, uniform refinement can lead to sub-optimal orders of convergence.
- most of the time we do not know the exact solution u but still need to estimate the discretization error.
- some parts of the domain may need to be refined more than others: doing uniform refinement may be a waste of computational time.

FEM: does it work ?

So what's the problem ?

- sometimes, uniform refinement can lead to sub-optimal orders of convergence.
- most of the time we do not know the exact solution u but still need to estimate the discretization error.
- some parts of the domain may need to be refined more than others: doing uniform refinement may be a waste of computational time.
- to do so we need a local estimation of the discretization error.

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A posteriori error estimation

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- get information about the local errors,
- keep the estimator computational cost at most of same complexity order than the computation of u_h.

A posteriori error estimation

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Although, this is almost never possible.

A posteriori error estimation

Instead, for most of a posteriori error estimators

$$\eta^2 := \sum_{T\in\mathcal{T}_h} \eta_T^2,$$

we can show existence of (unknown) constants c and C, only dependent of some regularity properties of the mesh (but independent of h and h_T) so that

$$c\eta_T \leqslant \|\nabla e\|_T, \quad \text{(local efficiency)}$$

and

$$\|\nabla e\| \leqslant C\eta. \quad \text{(global reliability)}$$

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Example: Bank-Weiser a posteriori error estimator

We can show that $e := u - u_h$ is the unique solution of

$$\int_\Omega \nabla e \cdot \nabla v = \sum_{T \in \mathcal{T}_h} \int_T (f - \Delta u_h) v + \sum_{E \in \mathcal{E}_h^I} \int_E \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right] v, \quad \forall v \in H_0^1(\Omega).$$

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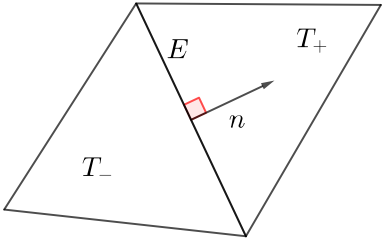

$$\frac{\partial v_h}{\partial n} := n \cdot \nabla v_h,$$

$$[\![v]\!] := v_{|T_-} - v_{|T_+}.$$

Example: Bank-Weiser a posteriori error estimator

Find e_T in $H_0^1(T)$ such that

$$\int_T \nabla e_T \cdot \nabla v_T = \int_T (f - \Delta u_h) v_T + \sum_{E \in \partial T \cap \mathcal{E}_h^I} \int_E \frac{1}{2} \left[\!\!\left[\frac{\partial u_h}{\partial n} \right]\!\!\right] v_T, \quad \forall v_T \in H_0^1(T).$$

Main idea: Discretize this problem using finite element methods!

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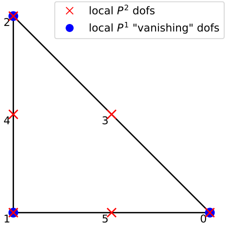

Example: Bank-Weiser a posteriori error estimator

Then, for each triangle T

Find $e_{BW,T}$ in V_T^{BW} such that

$$\begin{aligned}\int_T \nabla e_{BW,T} \cdot \nabla v_{BW,T} &= \int_T (f - \Delta u_h) v_{BW,T} \\ &+ \sum_{E \in \partial T \cap \mathcal{E}_h^I} \int_E \frac{1}{2} \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right] v_{BW,T}, \\ &\forall v_{BW,T} \in V_T^{BW}.\end{aligned}$$

Example: Bank-Weiser a posteriori error estimator

Definition of Bank-Weiser estimator: For a triangle T of the mesh, the Bank-Weiser estimator is defined by

$$\eta_T := \|\nabla e_{BW,T}\|_T.$$

The global Bank-Weiser estimator is defined by

$$\eta^2 := \sum_{T \in \mathcal{T}_h} \|\nabla e_{BW,T}\|_T^2.$$

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Adaptive refinement process

Given a tolerance $\varepsilon > 0$, we can use the BW estimator to write the following algorithm:

While $\eta > \varepsilon$ do:

Solve Compute the FE solution u_h.

Estimate Compute the local BW estimators η_T on each triangle of the mesh (parallelizable), as well as $\eta := \left(\sum_{T\in\mathcal{T}_h} \eta_T^2\right)^{1/2}$.

Mark Mark the triangles according to some criterion (e.g. the ones for which $\eta_T \geqslant \theta \max_{T\in\mathcal{T}_h}(\eta_T)$).

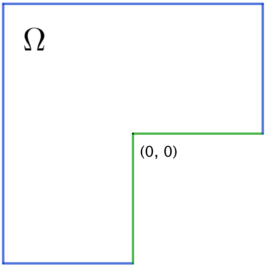
Refine Refine the marked triangles.

Return $\{\eta_T\}_T$, u_h.

Adaptive refinement process

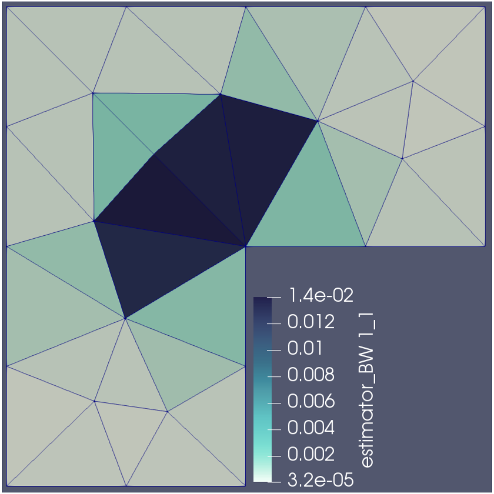
Let us apply this algorithm to our problematic test case:

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_0 \\ u = u_D \neq 0, & \text{on } \Gamma_1 \end{cases}$$

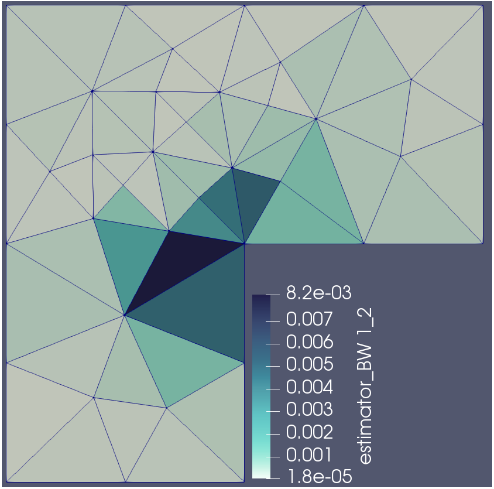


We discretise this equation with FEM of degree 1.

Adaptive refinement process



Adaptive refinement process



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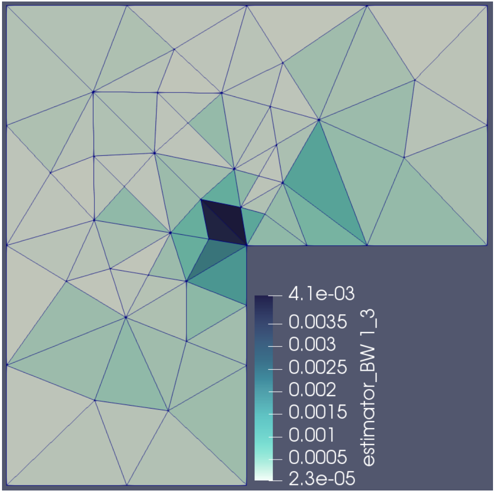

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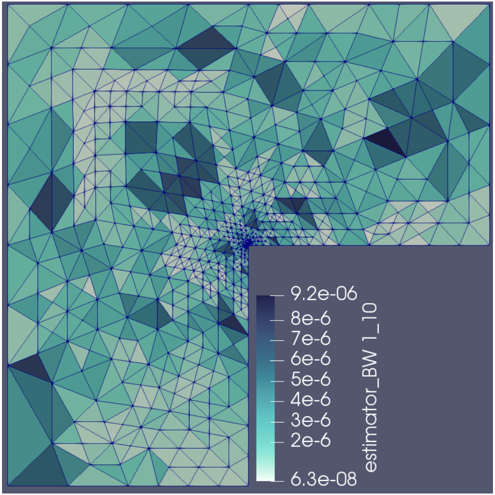

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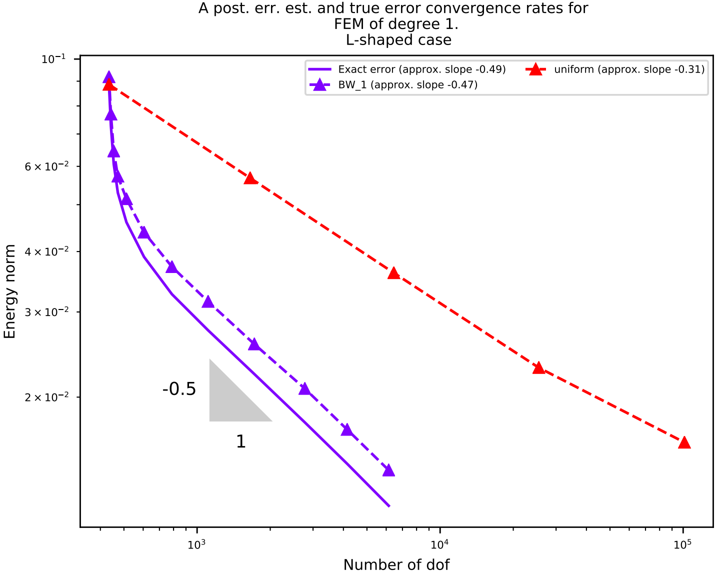

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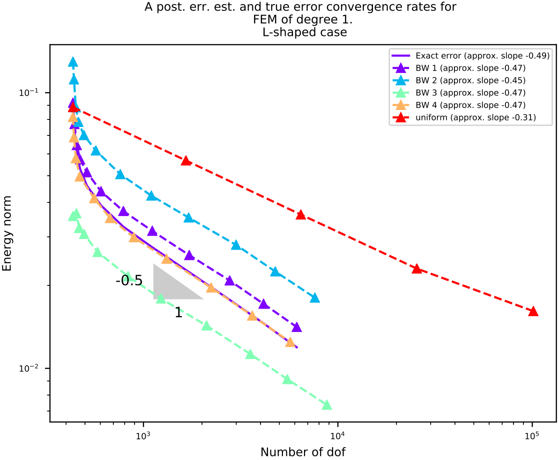
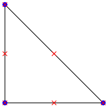
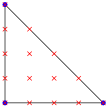


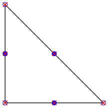
BW 1:



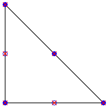
BW 2:



BW 3:



BW 4:



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Theory around Bank-Weiser estimator(s)

What we know:

Let u be the solution of Poisson's problem in a domain of dimension d and u_h be its finite element approximation of degree k.

Theorem: (Bank and Weiser [1985]) There exists a constant c only depending on the mesh regularity, independent of the mesh size, such that for any triangle T of the mesh

$$c\eta_T \leqslant \|\nabla e\|_T.$$

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Let u be the solution of Poisson's problem in a domain of dimension d and u_h be its finite element approximation of degree k.

Theorem: (Bank and Weiser [1985]) We assume in addition the (binding) *saturation hypothesis*. There exists a constant C only depending on the mesh regularity, independent of the mesh size such that

$$\|\nabla e\| \leqslant C\eta.$$

Theory around Bank-Weiser estimator(s)

What we know:

Let u be the solution of Poisson's problem in a domain of dimension d and u_h be its finite element approximation of degree k.

Theorem: (Nochetto [1993]) Let $d \leqslant 2,\ k = 1$ and $\mathcal{I}_T : V_T^2 \longrightarrow V_T^1$. Then, for the corresponding BW estimator, there exists a constant C only depending on the mesh regularity, independent of the mesh size such that

$$\|\nabla e\| \leqslant C\eta.$$

Theory around Bank-Weiser estimator(s)

What we know:

Let u be the solution of Poisson's problem in a domain of dimension d and u_h be its finite element approximation of degree k.

Theorem: (Verfürth [1994]) For each triangle T, we define $\mathcal{I}_T : V_T^{\overline{k}} \longrightarrow V_T^k$, with $\overline{k} \geqslant k+3$.
Then, for the corresponding BW estimator, there exists a constant C only depending on the mesh regularity, independent of the mesh size, such that

$$\|\nabla e\| \leqslant C\eta.$$

Theory around Bank-Weiser estimator(s)

What we know:

Let u be the solution of Poisson's problem in a domain of dimension $d = 2$.

Theorem: (from an idea of Morin et al. [2002]) Let $(\mathcal{T}_l)_l$ be a sequence of meshes constructed by successive adaptive refinements steered by the Bank-Weiser estimator defined from $\mathcal{I}_T : V_T^{k+1} \longrightarrow V_T^k$ and $V_T^{BW} := \ker(\mathcal{I}_T)$ and $(u_l)_l$ the corresponding finite element solutions of degree $k = 1$. Then there exist two constants $\alpha < 1$ and C depending only on the mesh regularity such that

$$\|\nabla(u - u_l)\| \leqslant C\alpha^l.$$

Theory around Bank-Weiser estimator(s)

What we do not know:

Let u be the solution of Poisson's problem in a domain of dimension d and u_h be its finite element approximation of degree k.

Theorem ? For each triangle T, we define $\mathcal{I}_T : V_T^{k+1} \longrightarrow V_T^k$, and $V_T^{BW} := \ker(\mathcal{I}_T)$.
Then, for the corresponding BW estimator, there exist a constant C only depending on the mesh regularity, independent of the mesh size, such that

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Ongoing/Future work

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- Compare different variants of Bank-Weiser with other estimators.
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- Apply Bank-Weiser estimation to elasticity problems.
- Integrate Bank-Weiser estimation to a Multi-level Monte Carlo algorithm for solving SPDEs.
- Find a proof of the last theorem ?

Thank you for you attention !

References I

Alnæs, M. S., Blechta, J., Hake, J., Johansson, A., Kehlet, B., Logg, A., Richardson, C., Ring, J., Rognes, M. E., and Wells, G. N. (2015). The FEniCS Project Version 1.5. *Arch. Numer. Softw.*, 3(100).

Bank, R. E. and Weiser, A. (1985). Some a posteriori error estimators for elliptic partial differential equations. *Math. Comp.*, 44(170):283–301.

Logg, A. (2007). Automating the Finite Element Method. *Arch. Comput. Methods Eng.*, 14(2):93–138.

Logg, A. and Wells, G. N. (2010). DOLFIN: Automated Finite Element Computing. *ACM Trans. Math. Softw.*, 37(2).

Morin, P., Nochetto, R. H., and Siebert, K. G. (2002). Local problems on stars: A posteriori error estimators, convergence, and performance. *Math. Comput.*, 72(243):1067–1098.

Nochetto, R. H. (1993). Removing the saturation assumption in a posteriori error analysis. *Istit. Lomb. Accad. Sci. Lett. Rend. A*, 127(1):67—-82 (1994).

Verfürth, R. (1994). A posteriori error estimation and adaptive mesh-refinement techniques. *J. Comput. Appl. Math.*, 50:67–83.